

MULTIPLICATIVITY OF PERVERSE FILTRATION IN HILBERT SCHEME OF FIBERED SURFACES

ZILI ZHANG

ABSTRACT. Let $S \rightarrow C$ be projective surface with numerically trivial canonical bundle fibered onto a curve. We prove the multiplicativity of perverse filtration on $H^*(S^{[n]}, \mathbb{Q})$ for the natural morphism $S^{[n]} \rightarrow C^{(n)}$. We also prove the multiplicativity for five families of Hitchin systems obtained in a similar way and compute the perverse numbers of the Hitchin moduli spaces. We show that the perverse numbers match the predictions of the numerical version of the de Cataldo-Hausel-Migliorini $P = W$ conjecture and of the conjecture by Hausel-Letellier-Villegas for small values of n .

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1. INTRODUCTION

There are natural moduli spaces associated to any smooth complex projective variety X and an algebraic reductive group G . The moduli of Higgs G -bundles M_D . Closely related are the character variety M_B , which is the moduli of representations of fundamental group in G . In [15], Simpson proved that these moduli spaces are algebraic varieties, and they are canonically diffeomorphic to each other. The diffeomorphism induces canonical identification of cohomology of these moduli spaces. These moduli spaces are algebraic varieties, so the cohomologies are naturally endowed with mixed Hodge structures. Furthermore, the moduli of Higgs bundle carries a proper Hitchin map to an affine space, so $H^*(M_D)$ is endowed with the Leray filtration and the perverse (Leray) filtration. Under the canonical isomorphism, one may compare the filtrations mentioned above. One remarkable result is in [5], where M. de Cataldo, T. Hausel and L. Migliorini considered the

case when X is any curve with genus $g \geq 2$ and $G = GL(2, \mathbb{C})$. They proved that, via the non-abelian Hodge theorem, the perverse filtration P on $H^*(M_D)$ for the Hitchin map equals the mixed Hodge theoretic weight filtration W on $H^*(M_B)$, they proved that $P = W$. Moreover, there are numerous and diverse Hitchin-type moduli spaces M_D that come with natural Hitchin-type maps $h : M_D \rightarrow \mathbb{A}$ and which have corresponding (twisted) character varieties M_B . The $P = W$ conjecture predicts that under the canonical isomorphism between the cohomology groups, the perverse filtration and the weight filtration correspond.

In this paper, we consider a beautiful and classical class of Hitchin systems $h : M_D \rightarrow \mathbb{A}$. They are five families of moduli spaces of parabolic Higgs bundles over \mathbb{P}^1 with marked points, labeled by affine Dynkin diagram $\widetilde{A}_0, \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7$ and \widetilde{E}_8 . In this setting, a result of M. Gröchenig in [9] states that these M_D are the Hilbert schemes $S^{[n]}$ of n -points of five distinct smooth algebraic elliptically fibered surfaces $S \rightarrow \mathbb{A}^1$. There are, for each of the five surfaces and for each $n \geq 1$, Hitchin-type maps $h : M_D = S^{[n]} \rightarrow \mathbb{A}^n$, hence a perverse filtration P on the cohomology groups $H^*(M_D)$.

Multiplicativity of perverse filtration

The Leray filtration in cohomology is multiplicative for the cup product, which means $\cup : L_i H^p \times L_j H^q \rightarrow L_{i+j} H^{p+q}$. The perverse filtration is *not* in general, even for proper maps between smooth projective varieties. Since it had been known previously that W is multiplicative, one key step in the proof of $P = W$ in [5] was to establish that P is also multiplicative. These surfaces, and their Hilbert schemes M_D have trivial canonical bundle. The construction of Hitchin-type map $h : M_D \rightarrow \mathbb{A}$ is analogous to the one that starts with an elliptic K3 (which has trivial canonical bundle) $S \rightarrow \mathbb{P}^1$ and yields the natural map $h : S^{[n]} \rightarrow \mathbb{P}^n$. In fact, one can start with any compact surface with numerically trivial canonical bundle $S \rightarrow C$ fibered over a curve and obtain a map $h : S^{[n]} \rightarrow C^{(n)}$ (n -th symmetric product of C); however, in this generality, there is no Hitchin-Higgs-type interpretation for h . It is natural to ask whether the perverse filtrations for all these maps h are multiplicative. The main result of this paper is as follows. Let h be any of the maps above.

Theorem. *The perverse filtration on $H^*(S^{[n]}, \mathbb{Q})$ of the map h is multiplicative.*

In the proof, we use the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [1], as well as the associated “relative” Hodge-Riemann bilinear relations, due to M. de Cataldo and L. Migliorini [3], to “calculate” the perverse filtration and to produce special basis for the cohomology groups which are adapted to our problem. The key step is the determination of the precise perversity of the class of the small diagonals in the products S^n ; the general bounds for these perversities are too weak for the problem at hand, and we had to improve upon them by using the special geometry. The description in of M. Lehn [13], of the cohomology ring of $H^*(S^{[n]})$ is a key ingredient in our approach.

Numerical $P = W$

There is a numerical version of $P = W$ conjecture, namely instead of requiring the filtrations to correspond, one only requires the dimensions of graded pieces

to be the same. Conjectures in [6] and [10] predict the perverse numbers and mixed Hodge numbers in general. In our five families of Hitchin systems, We computed the perverse filtration explicitly, as can the perverse numbers. Let $P_n(q, t) = \sum_{i,j} \dim \operatorname{Gr}_i H^j(M_D) q^i t^j$ be the perverse Poincaré polynomial.

Theorem. *For \widetilde{A}_0 case, the generating series for the perverse Poincaré polynomial is*

$$\sum_{n=0}^{\infty} s^n P_n(q, t) = \prod_{m=1}^{\infty} \frac{(1 + s^m q^m t^{2m-1})^2}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^{m+1} t^{2m})}.$$

For the other four cases, the generating series are

$$\sum_{n=0}^{\infty} s^n P_n(q, t) = \prod_{m=1}^{\infty} \frac{1}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^m t^{2m})^k (1 - s^m q^{m+1} t^{2m})}$$

where k is an integer determined explicitly by the geometry of map f .

Using the explicit description of the corresponding character varieties for $n = 1$ in [8], we prove the full version of $P = W$ conjecture for $S \rightarrow \mathbb{A}^1$ in each of our five cases. For $n \geq 2$, the corresponding character varieties are unknown yet. However, there are conjectures concerning the shape of the filtration W on $H^*(M_B)$ in [10]. Mathematica computations show that for small n , the perverse numbers obtained in our theorem match the conjectural mixed Hodge numbers.

The paper is organized as follows. In section 2, for any map $f : X \rightarrow Y$ between smooth projective varieties, we use a weak version of relative Hodge Riemann bilinear relation to pick up “orthonormal” basis of $H^*(X)$ with perversity controls. In section 3, we write down a perverse decomposition to prove that the perversity is additive with respect to tensor product. Assuming the perversity filtration associated to each factor is multiplicative, we use the additivity and the basis in section 2 to estimate the perversity of small diagonal in the product of the map. In section 4, we show that the symmetric product and the alternating product of perverse sheaf is perverse. In section 5, we give a decomposition of the symmetric product and prove that it is compatible with the one for the cartesian product. In section 6, we focus on the case that smooth surfaces S properly maps to smooth curve C . We use the decomposition theorem for Hilbert-Chow morphism to find a perverse decomposition for Hitchin-type map $h : S^{[n]} \rightarrow C^{(n)}$. In particular, we compute the perverse filtration of h . In section 7, we follow Lehn’s work [13] to state the ring structure of $H^*(S^{[n]})$ where S is a surface with numerically trivial canonical bundle. In section 8, we prove that multiplicativity for perverse filtration for the map from smooth surface to smooth curve by comparing the perverse filtration with Leray filtration. We use perversity of diagonal to prove the push-forward lemma 8.6, and use Lehn’s formula to prove that the perverse filtration on $H^*(S^{[n]})$ is multiplicative. In section 9, we deal with five types of Hitchin system, which can be realized as Hilbert scheme of points on smooth surfaces. We modify the method in compact case to get the same estimation of the perversity of diagonal, and conclude that the perverse filtration is multiplicative. In section 10, we prove a full version of $P = W$ conjecture for $n = 1$ case defined in section 9. In section 11, we compute the dimension of graded piece of perverse filtration, and state a numerical $P = W$

conjecture for our five types of Hitchin system, which is verified for small values of n .

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2. PERVERSE FILTRATION AND BASIS FOR THE COHOMOLOGY

In this section we recall the basic properties of perverse filtration associated to maps between algebraic varieties. We also choose a basis for the cohomology group with certain properties, which is crucial in the perversity estimation of the diagonal embedding.

We always work on varieties over \mathbb{C} . All cohomology groups have \mathbb{Q} coefficients.

Definition 2.1. Let $f : X \rightarrow Y$ be a morphism between smooth projective varieties. Let $r(f) = \dim X \times_Y X - \dim X$ be the defect of semismallness. Define the geometric perverse filtration as

$$P_p H^d(X; \mathbb{Q}) = \text{Im} \left\{ \mathbb{H}^{d-\dim X+r(f)}(Y, {}^p\tau_{\leq p} Rf_* \mathbb{Q}_X[\dim X - r(f)]) \rightarrow H^d(X, \mathbb{Q}) \right\},$$

where the ${}^p\tau_{\leq p}$ is the truncation functor of perverse t -structure. The filtration ranges from 0 to $2r(f)$, i.e. $\text{Gr}_p^P H^d(X) = 0$ if $p < 0$ or $p > 2r(f)$.

Definition 2.2. Let $f : X \rightarrow Y$ as before. Given a cohomology class $0 \neq \alpha \in H^*(X)$, define the perversity of α , denoted as $\mathfrak{p}(\alpha)$, to be the integer that $\alpha \in P_{\mathfrak{p}(\alpha)} H^*(X)$ and $\alpha \notin P_{\mathfrak{p}(\alpha)-1} H^*(X)$. By our choice of perversity, function \mathfrak{p} takes value in the interval $[0, 2r(f)]$. Define $\mathfrak{p}(0) = -\infty$.

Remark 2.3. Under the notation above, the perverse filtration is multiplicative with respect to cup product can be rephrased as for any class $\alpha, \beta \in H^*(X)$, $\mathfrak{p}(\alpha \cup \beta) \leq \mathfrak{p}(\alpha) + \mathfrak{p}(\beta)$.

The remaining part of this section is devoted to prove the following proposition. A basis with following property is crucial to control the change of the perversity under diagonal embedding.

Proposition 2.4. Let $f : X \rightarrow Y$ as above. Let $k(p, d) = \dim \text{Gr}_p^P H^d(X)$. There exists additive \mathbb{Q} -basis

$$B = \{\beta_{p,i}^d \mid 0 \leq d \leq 2 \dim X, 0 \leq p \leq 2r(f), 1 \leq i \leq k(p, d)\} \subset H^*(X)$$

with the following property.

- (1) $\beta_{p,i}^d \in P_p H^d(X)$. $\{\overline{\beta_{p,1}^d}, \dots, \overline{\beta_{p,k(p,d)}^d}\}$ is a basis of $\text{Gr}_p^P H^d(X)$, where $\overline{\beta_{p,i}^d}$ is the image of $\beta_{p,i}^d$ under the natural quotient map $P_p H^d(X) \rightarrow \text{Gr}_p^P H^d(X)$.
- (2) The basis B is signed orthonormal in the following sense.

$$\langle \beta_{p,i}^d, \beta_{p',j}^{d'} \rangle = \begin{cases} \pm 1 & d + d' = 2 \dim X, p + p' = 2r(f) \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

where -1 can only appear in the case $d = d' = \dim X$, $p = p' = r(f)$ and $i = j$.

In particular, if $A = \{\alpha_{p,i}^d\}$ is the dual basis of B with respect to Poincaré pairing. (Indices for α are only for indexing, which has no perversity or degree meaning in priori.) Then we have

$$\mathfrak{p}(\alpha_{p,i}^d) + \mathfrak{p}(\beta_{p,i}^d) = 2r(f)$$

To prove proposition 2.4, we need the following two results.

Lemma 2.5 (de Cataldo, Migliorini). *Let $f : X \rightarrow Y$ be a morphism of algebraic varieties. The Poincaré pairing*

$$P_p H^d(X) \times P_{p'} H^{2 \dim X - d}(X) \rightarrow \mathbb{Q}$$

is trivial for $p + p' < 2r(f)$.

Proof. Denote by $D(Y)$ the derived category of constructible sheaf on Y . Let $\epsilon : Rf_* \mathbb{Q}_X[n] \rightarrow \mathcal{D}(Rf_* \mathbb{Q}_X[n])$ be the duality isomorphism. For every d , the map ϵ defines the non-degenerate Poincaré pairing

$$\int_X : H^d(X) \times H^{2 \dim X - d}(X) \rightarrow \mathbb{Q}.$$

So to prove the pairing is trivial, it suffices to prove that the following composition is 0:

$${}^p\tau_{\leq p-r(f)} Rf_* \mathbb{Q}_X[n] \rightarrow Rf_* \mathbb{Q}_X[n] \xrightarrow{\epsilon} \mathcal{D}(Rf_* \mathbb{Q}_X[n]) \rightarrow \mathcal{D}({}^p\tau_{\leq p'-r(f)} Rf_* \mathbb{Q}_X[n]).$$

By our choice of geometric perversity, the dualizing functor \mathcal{D} satisfies

$$\mathcal{D}\left(D(Y)^{\leq p'-r(f)}\right) \subset D(Y)^{\geq r(f)-p'}.$$

By axiom of t -structure, $\text{Hom}(D(Y)^{\leq p-r(f)}, D(Y)^{\geq r(f)-p'}) = 0$, since $p - r(f) < r(f) - p'$. So the composition is 0. \square

Lemma 2.6. *The pairing induced by Poincaré pairing*

$$\text{Gr}_p^P H^d(X) \times \text{Gr}_{2r(f)-p}^P H^{2n-d}(X) \rightarrow \mathbb{Q}$$

is non-degenerate.

Proof. By lemma 2.5, the pairing is well-defined. The non-degeneracy is due to Theorem 2.1.4 and corollary 2.1.8 of [3] \square

Proof of proposition 2.4. Denote $B_p^d = \{\beta \in B \mid \mathfrak{p}(\beta) = p, \beta \in H^d(X)\}$. We construct B_p^d using Gram-Schmidt type argument. We proceed the construction inductively in the lexicographical order of pairs (p, d) .

Induction base: for $(p, d) \prec (r(f), \dim X)$, pick any basis of $\text{Gr}_p^P H^d(X)$, and lift them to get B_p^d . For $(p, d) = (r(f), \dim X)$, by lemma 2.6, the self-intersection form is nondegenerate, so we may pick a basis such that the intersection matrix is diagonal and only have ± 1 on diagonal. Denote any lift of this basis by $B_{r(f)}^{\dim X}$.

Now we are going to find $B_{p,d} = \{\beta_{p,1}^d, \dots, \beta_{p,k(p,d)}^d\}$, assuming that all cases precede to (p, d) are done. To simplify notation, we let $e = 2 \dim X - d$ and $q = 2r(f) - p$. Note that $(q, e) \prec (p, d)$. First by lemma 2.5, pairing $\text{Gr}_p^P H^d(X) \times \text{Gr}_q^P H^e(X)$

is nondegenerate, so we may pick basis $\widetilde{B}_p^d = \{\widetilde{\beta}_{p,1}^d, \dots, \widetilde{\beta}_{p,k(p,d)}^d\}$ such that the

matrix of this bilinear pairing is identity matrix under basis B_q^e and \widetilde{B}_p^d . Modify the basis by setting

$$(2.7) \quad \begin{pmatrix} \beta_{p,1}^d \\ \cdots \\ \beta_{p,k(p,d)}^d \end{pmatrix} = \begin{pmatrix} \widetilde{\beta_{p,1}^d} \\ \cdots \\ \widetilde{\beta_{p,k(p,d)}^d} \end{pmatrix} + \sum_{i=q+1}^{p-1} A_i \begin{pmatrix} \beta_{i,1}^d \\ \cdots \\ \beta_{i,k(i,d)}^d \end{pmatrix},$$

where A_i is a $k(p,d) \times k(i,d)$ matrix of rational numbers to be determined. The condition to that A_i need to satisfy is slightly different when $d < \dim X$, $d > \dim X$ and $d = \dim X$.

- (1) $d < \dim X$. By degree reason, it suffices to require the orthogonality between B_p^d and degree d basis which precedes (p,d) in lexicographical order, namely B_1^e, \dots, B_{p-1}^e . If we denote the Poincaré pairing by regular multiplication, then the condition can be written in of matrix, namely

$$\begin{pmatrix} \beta_{p,1}^d \\ \cdots \\ \beta_{p,k(p,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{j,1}^e & \cdots & \beta_{j,k(j,e)}^e \end{pmatrix} = 0$$

for $j = 0, \dots, \hat{q}, \dots, i-1$, and

$$\begin{pmatrix} \beta_{p,1}^d \\ \cdots \\ \beta_{p,k(p,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{q,1}^e & \cdots & \beta_{q,k(q,e)}^e \end{pmatrix} = I_{k(p,d)},$$

where I denotes the identity matrix. Plug in (2.7), we have

$$\begin{pmatrix} \widetilde{\beta_{p,1}^d} \\ \cdots \\ \widetilde{\beta_{p,k(p,d)}^d} \end{pmatrix} \begin{pmatrix} \beta_{j,1}^e & \cdots & \beta_{j,k(j,e)}^e \end{pmatrix} + \sum_{i=q+1}^{p-1} A_i \begin{pmatrix} \beta_{i,1}^d \\ \cdots \\ \beta_{i,k(i,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{j,1}^e & \cdots & \beta_{j,k(j,e)}^e \end{pmatrix} = 0$$

for $j = 0, \dots, \hat{q}, \dots, p-1$, and

$$\begin{pmatrix} \widetilde{\beta_{p,1}^d} \\ \cdots \\ \widetilde{\beta_{p,k(p,d)}^d} \end{pmatrix} \begin{pmatrix} \beta_{q,1}^e & \cdots & \beta_{q,k(q,e)}^e \end{pmatrix} + \sum_{i=q+1}^{p-1} A_i \begin{pmatrix} \beta_{i,1}^d \\ \cdots \\ \beta_{i,k(i,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{q,1}^e & \cdots & \beta_{q,k(q,e)}^e \end{pmatrix} = I$$

The second condition is always satisfied by $q+i < 2r(f)$ and lemma 2.5. The first condition is true when $j < q$ by the same reason. When $q \leq j \leq p-1$, by induction hypothesis, the first condition is reduced to

$$\begin{pmatrix} \widetilde{\beta_{p,1}^d} \\ \cdots \\ \widetilde{\beta_{p,k(p,d)}^d} \end{pmatrix} \begin{pmatrix} \beta_{j,1}^e & \cdots & \beta_{j,k(j,e)}^e \end{pmatrix} + A_{2r(f)-j} = 0$$

This solves $A_{2r(f)-j}$. Note that $q+1 \leq j \leq p-1$, so $q+1 \leq 2r(f)-j \leq p-1$ (note that $p+q = 2r(f)$). That means that all A_i are determined.

- (2) $j > \dim X$. The only difference in this case is that B_p^e is already done, so we need one more condition to require B_p^d to be orthogonal to B_p^e . To make this work, the sum taken in (2.7) need to be add from q to $p-1$ instead of from $q+1$ to $p-1$. The computation are similar.

- (3) $j = \dim X$. In this case $\widetilde{B_p^{\dim X}}$ need to be modified to be isotropical. The condition to be satisfied is exactly the same as $j > \dim X$ case, the result is slightly different: the matrix A_q is differed by a factor 2.

This completes the induction. In particular, dual basis $\alpha_{p,i}^d = \pm \beta_{q,i}^e$, so $\mathfrak{p}(\alpha_{p,i}^d) + \mathfrak{p}(\beta_{q,i}^e) = 2r(f)$. \square

Remark 2.8. The assumption that X and Y are smooth varieties is not necessary. In fact the construction works for the intersection cohomology for singular varieties.

Remark 2.9. We point out an easy but important fact about B . The basis B is filtered in the sense that

$$P_p H^*(X) = \text{Span } \{\beta \in B \mid \mathfrak{p}(\beta) \leq p\}.$$

3. PERVERSE FILTRATION OF PRODUCT AND PERVERSITY OF DIAGONAL

In this section we show that perversity is additive for the tensor product. We also study how the perversity changes under the Gysin push-forward along diagonal embedding.

Proposition 3.1. *Let $f_1 : X_1 \rightarrow Y_1$, $f_2 : X_2 \rightarrow Y_2$ be two proper morphisms between smooth quasi-projective varieties. Let*

$$Rf_{1,*} \mathbb{Q}_{X_1}[\dim X_1 - r(f_1)] \cong \bigoplus_{i=0}^{2r(f_1)} \mathcal{F}_i[-i],$$

$$Rf_{2,*} \mathbb{Q}_{X_2}[\dim X_2 - r(f_2)] \cong \bigoplus_{j=0}^{2r(f_2)} \mathcal{G}_j[-j]$$

be corresponding perverse decomposition, where \mathcal{F}_i and \mathcal{G}_j are perverse sheaves. Then

$$R(f_1 \times f_2)_* \mathbb{Q}_{X_1 \times X_2}[\dim X_1 \times X_2 - r(f_1 \times f_2)] \cong \bigoplus_{i,j} \mathcal{F}_i \boxtimes \mathcal{G}_j[-i-j]$$

is a perverse decomposition of proper map $p \times q : X_1 \times X_2 \rightarrow Y_1 \times Y_2$. In particular, for $\alpha_1 \in H^(X_1)$, $\alpha_2 \in H^*(X_2)$, we have $\mathfrak{p}(\alpha_1 \otimes \alpha_2) = \mathfrak{p}(\alpha_1) + \mathfrak{p}(\alpha_2)$, where $\alpha_1 \otimes \alpha_2$ is viewed as a cohomology class in $H^*(X_1 \times X_2)$, and the perversity is taken with respect to map $f_1 \times f_2$.*

Proof. Note that f_1 , f_2 and $f_1 \times f_2$ are all proper, so $Rf_* = Rf_!$. By the Künneth formula (see exercise II.18 of [12]), we have

$$\begin{aligned} & R(f_1 \times f_2)_* \mathbb{Q}_{X_1 \times X_2}[\dim X_1 + \dim X_2 - r(f_1) - r(f_2)] \\ &= R(f_1 \times f_2)_* \mathbb{Q}_{X_1} \boxtimes \mathbb{Q}_{X_2}[\dim X_1 + \dim X_2 - r(f_1) - r(f_2)] \\ &= Rf_{1,*} \mathbb{Q}_{X_1}[\dim X_1 - r(f_1)] \boxtimes Rf_{2,*} \mathbb{Q}_{X_2}[\dim X_2 - r(f_2)] \\ &\cong \bigoplus_{i,j} \mathcal{F}_i \boxtimes \mathcal{G}_j[-i-j] \end{aligned}$$

By proposition 10.3.6 (i)(ii) of [12], the box product $\mathcal{F}_i \boxtimes \mathcal{G}_j$ is perverse. Therefore this gives a perverse decomposition. To check the perversity is additive with respect to tensor product is basically by definition. Let $\mathfrak{p}(\alpha_1) = p_1$, $\mathfrak{p}(\alpha_2) = p_2$. If $\alpha_1 = 0$

or $\alpha_2 = 0$, the equation holds trivially. Suppose none of them are 0, recall that by definition of geometric perversity, we have

$$\alpha_1 \in \mathbb{H} \left(\bigoplus_{i \leq p_1} \mathcal{F}_i[-i] \right), \alpha_2 \in \mathbb{H} \left(\bigoplus_{j \leq p_2} \mathcal{G}_j[-j] \right)$$

So

$$\begin{aligned} \alpha_1 \otimes \alpha_2 &\in \mathbb{H} \left(\bigoplus_{i \leq p_1, j \leq p_2} \mathcal{F}_i \boxtimes \mathcal{G}_j[-i-j] \right) \\ &\subset \mathbb{H} \left(\bigoplus_{i+j \leq p_1+p_2} \mathcal{F}_i \boxtimes \mathcal{G}_j[-i-j] \right) \\ &= P_{p_1+p_2} H^*(X_1 \times X_2) \end{aligned}$$

This shows that $\mathbf{p}(\alpha_1 \otimes \alpha_2) \leq \mathbf{p}(\alpha_1) + \mathbf{p}(\alpha_2)$. On the other hand, $\mathbf{p}(\alpha_k) = p_k$ means that $\alpha_k \neq 0 \in \mathrm{Gr}_{p_k}^P H^*(X_k)$. So

$$\alpha_1 \otimes \alpha_2 \neq 0 \in \mathrm{Gr}_{p_1}^P H^*(X_1) \otimes \mathrm{Gr}_{p_2}^P H^*(X_2) \subset \mathrm{Gr}_{p_1+p_2}^P H^*(X_1 \times X_2)$$

This shows that $\mathbf{p}(\alpha_1 \otimes \alpha_2) = \mathbf{p}(\alpha_1) + \mathbf{p}(\alpha_2)$. \square

Corollary 3.2. *Let $f : X \rightarrow Y$ be a morphism between smooth projective varieties. Let $B = \{\beta_1, \dots, \beta_k\}$ be the basis of $H^*(X)$ in proposition 2.4. Then the B^n defined by*

$$B^n := \{\beta_{i_1} \otimes \dots \otimes \beta_{i_n} \mid 1 \leq i_1, \dots, i_n \leq k\}$$

is a basis of $H^(X^n)$. Furthermore, this basis is filtered with respect to the perverse filtration induced by map $f^n : X^n \rightarrow Y^n$.*

Proposition 3.3. *Let $f : X \rightarrow Y$ be any morphism between smooth projective varieties. Suppose the perverse filtration for $f : X \rightarrow Y$ is multiplicative, i.e. for any class $\alpha_1, \alpha_2 \in H^*(X)$, we have $\mathbf{p}(\alpha_1 \cup \alpha_2) \leq \mathbf{p}(\alpha_1) + \mathbf{p}(\alpha_2)$. The small diagonal embedding $\Delta_n : X \rightarrow X^n$ induces a Gysin push-forward of cohomology*

$$\Delta_{n,*} : H^*(X) \rightarrow H^{*+2(n-1)\dim X}(X)$$

Then for any $\gamma \in H^(X)$, we have the perversity of diagonal*

$$\mathbf{p}(\Delta_{n,*}(\gamma)) \leq \mathbf{p}(\gamma) + 2(n-1)r(f),$$

where the perversity is defined with respect to the map $f^n : X^n \rightarrow Y^n$.

We need an easy fact to prove the proposition.

Lemma 3.4. *Let X be a compact smooth manifold. Let β_1, \dots, β_k be an additive \mathbb{Q} -basis of $H^*(X)$. Let $\alpha_1, \dots, \alpha_k$ be the dual basis in the sense of Poincaré pairing, namely $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. Then*

$$\Delta_{2,*}(\gamma) = \sum_{i=1}^k \alpha_i \otimes (\beta_i \cup \gamma)$$

Proof. Let $\text{pr}_1, \text{pr}_2 : X \times X \rightarrow X$ be the projection map to two factors. Any cohomology class $\Phi \in H^*(X \times X)$ induces a correspondence

$$\begin{aligned} [\Phi] : H^*(X) &\rightarrow H^*(X) \\ \xi &\mapsto \text{pr}_{2,*}(\text{pr}_1^*(\xi) \cup \Phi) \end{aligned}$$

Now the correspondence induced by left hand side is

$$\begin{aligned} [\Delta_{2,*}(\gamma)](\xi) &= \text{pr}_{2,*}(\text{pr}_1^*(\xi) \cup \Delta_{2,*}(\gamma)) \\ &= \text{pr}_{2,*}(\xi \otimes 1 \cup \Delta_{2,*}(\gamma)) \\ &= \text{pr}_{2,*} \Delta_{2,*}(\Delta_2^*(\xi \otimes 1) \cup \gamma) \\ &= \Delta_2^*(\xi \otimes 1) \cup \gamma \\ &= \xi \cup \gamma, \end{aligned}$$

where the second equality is due to projective formula, and the third equality uses $\Delta_2 \circ \text{pr}_2 = \text{id}$. The correspondence on right hand side computes as

$$\begin{aligned} \left[\sum_{i=1}^k \alpha_i \otimes \beta_i \cup \gamma \right](\beta_j) &= \sum_{i=1}^k \text{pr}_{2,*}(\beta_j \cup \alpha_i \otimes \beta_i \cup \gamma) \\ &= \text{pr}_{2,*}(\beta_j \cup \alpha_j \otimes \beta_j \cup \gamma) \\ &= \beta_j \cup \gamma \end{aligned}$$

Here we use the fact that the nontrivial push-forward takes place only when $\beta_j \cup \alpha_i$ is a cohomology class of top degree, and hence in this case $\beta_j \cup \alpha_i = \langle \beta_j, \alpha_i \rangle = \delta_{ij}$ by our choice of $\{\alpha_i\}$ and $\{\beta_i\}$. So there is only one non-zero pairing survives in the summation. Now extend by linearity, we have

$$\left[\sum_{i=1}^k \alpha_i \otimes \beta_i \cup \gamma \right](\xi) = \xi \cup \gamma$$

So the lemma follows. \square

Proof of proposition 3.3. We use induction on n to prove the statement. Since $n = 1$ is trivial, we prove for $n = 2$ as induction basis.

Let $\{\beta_{p,i}^d\}, \{\alpha_{p,i}^d\}$ be the basis in proposition 2.4. Then by lemma 3.4 we have

$$\Delta_{2,*}(\gamma) = \sum_{p,d,i} \alpha_{p,i}^d \otimes (\beta_{p,i}^d \cup \gamma)$$

Now by proposition 3.1, proposition 2.4, hypothesis that multiplicativity of perverse filtration for $f : X \rightarrow Y$, we have

$$\begin{aligned} \mathbf{p}(\Delta_{2,*}(\alpha)) &\leq \max_{p,d,i} \mathbf{p}(\alpha_{p,i}^d \otimes (\beta_{p,i}^d \cup \gamma)) \\ &\leq \max_{p,d,i} (\mathbf{p}(\alpha_{p,i}^d) + \mathbf{p}(\beta_{p,i}^d) + \mathbf{p}(\gamma)) \\ &\leq 2r(f) + \mathbf{p}(\gamma) \end{aligned}$$

For general n , Δ_n can be decomposed into the following two diagonal maps.

$$X \xrightarrow{\Delta_{n-1}} X^{n-1} \xrightarrow{\Delta_2 \times \text{Id}^{n-2}} X^n$$

Then by induction hypothesis, we have

$$\begin{aligned} \mathbf{p}(\Delta_{n,*}(\gamma)) &\leq \mathbf{p}(\Delta_{n-1,*}\gamma) + 2r(f) \\ &\leq \mathbf{p}(\gamma) + 2(n-2)r(f) + 2r(f) \\ &= \mathbf{p}(\gamma) + 2(n-1)r(f) \end{aligned}$$

□

4. SYMMETRIC PRODUCT AND ALTERNATING PRODUCT OF PERVERSE SHEAVES

In this section, we give explicit description of the \mathfrak{S}_n action on n -fold external product of a bounded complex. We use the action to define symmetric product in derived category of constructible sheaves, and show that the symmetric product of perverse sheaf is still perverse. Our method is similar to the one in [14]. Let $X^{(n)} = X^n / \mathfrak{S}_n$ denote the n -th symmetric product of X .

Definition 4.1. Let K_i^\bullet be a bounded complex of constructible sheaves on complex quasi-projective variety X_i , for $1 \leq i \leq n$. Then the n -fold external product $\boxtimes_{i=1}^n K_i^\bullet$ on $\prod_{i=1}^n X_i$ is defined as the following.

- (1) The j -th component is $\bigoplus_{\sum p_i=j} \boxtimes_{i=1}^n K_i^{p_i}$.
- (2) Differential is $\sum_{i=1}^n (-1)^{p_1+\dots+p_{i-1}} d_i$ on the summand $\boxtimes_{i=1}^n K_i^{p_i}$, where d_i is induced by the differential of K_i .

Definition 4.2. Let K_i^\bullet and X_i as above. Then there is a natural \mathfrak{S}_n -action on $\boxtimes_{i=1}^n K_i^\bullet$ by:

$$\sigma^\# : \boxtimes_{i=1}^n K_i^\bullet \xrightarrow{\sim} \sigma_* \left(\boxtimes_{i=1}^n K_{\sigma(i)}^\bullet \right)$$

which is defined, for $m_i \in K_i^{p_i}$, by

$$\boxtimes_{i=1}^n m_i \mapsto (-1)^{\nu(\sigma,p)} \sigma_* \left(\boxtimes_{i=1}^n m_{\sigma(i)} \right)$$

where $\nu(\sigma, p) = \sum_{i < j, \sigma(j) < \sigma(i)} p_i p_j$.

Definition 4.3. Let X be a complex quasi-projective variety. Let $q : X^n \rightarrow X^{(n)}$ be the quotient map. For bounded complex of constructible sheaves K on X , we define symmetric product and alternating product as

$$\begin{aligned} K^{(n)} &= (Rq_* K^{\boxtimes n})^{\mathfrak{S}_n} \\ K^{\{n\}} &= (Rq_* K^{\boxtimes n})^{\text{sign}-\mathfrak{S}_n} \end{aligned}$$

where $(-)^{\mathfrak{S}_n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} Rq_*(\sigma^\#)$ is the symmetrizing projector and $(-)^{\text{sign}-\mathfrak{S}_n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{sign}(\sigma)} Rq_*(\sigma^\#)$ is the alternating projector. Here we use the fact that \mathfrak{S}_n acts trivially on $X^{(n)}$.

Remark 4.4. Under the definition, we have the canonical isomorphisms.

$$H^*(X^{(n)}, K^{(n)}) = H^*(X^n, K^{\boxtimes n})^{\mathfrak{S}_n} = \text{Sym}^* H^{\text{even}}(X, K) \otimes \wedge^* H^{\text{odd}}(X, K)$$

$$H^*(X^{(n)}, K^{\{n\}}) = H^*(X^n, K^{\boxtimes n})^{\text{sign}-\mathfrak{S}_n} = \wedge^* H^{\text{even}}(X, K) \otimes \text{Sym}^* H^{\text{odd}}(X, K)$$

Furthermore, we have

$$(K[a])^{(n)} = \begin{cases} K^{(n)}[na] & \text{if } a \text{ is even.} \\ K^{\{n\}}[na] & \text{if } a \text{ is odd.} \end{cases}$$

Proposition 4.5. Let \mathcal{P} be a perverse sheaf on X . Let $q : X^n \rightarrow X^{(n)}$ be the quotient map. Then $\mathcal{P}^{(n)}$ and $\mathcal{P}^{\{n\}}$ are perverse sheaves on $X^{(n)}$.

Proof. By proposition 3.1, $\mathcal{P}^{\boxtimes n}$ is perverse. Note that the map $q : X^n \rightarrow X^{(n)}$ is small, $Rq_* K^{\boxtimes n}$ is perverse. It suffices to prove that the invariant part and alternating part under \mathfrak{S}_n action are both perverse. By the definition of projector, we have

$$\begin{aligned} (Rq_* \mathcal{P}^{\boxtimes n})^{\mathfrak{S}_n} &\rightarrow Rq_* \mathcal{P}^{\boxtimes n} \rightarrow (Rq_* \mathcal{P}^{\boxtimes n})^{\mathfrak{S}_n} \\ (Rq_* \mathcal{P}^{\boxtimes n})^{\text{sign}-\mathfrak{S}_n} &\rightarrow Rq_* \mathcal{P}^{\boxtimes n} \rightarrow (Rq_* \mathcal{P}^{\boxtimes n})^{\text{sign}-\mathfrak{S}_n} \end{aligned}$$

where the compositions are identities. This means that $(Rq_* \mathcal{P}^{\boxtimes n})^{\mathfrak{S}_n}$ and $(Rq_* \mathcal{P}^{\boxtimes n})^{\text{sign}-\mathfrak{S}_n}$ are both direct summands of $Rq_* \mathcal{P}^{\boxtimes n}$ in the bounded derived category of constructible sheaves. The proposition holds due to the following lemma. \square

Lemma 4.6. *Let \mathcal{P} be a perverse sheaf on X . Let $\mathcal{P} = K \oplus K'$ where $K, K' \in D_c^b(X)$. Then K is perverse.*

Proof. In fact the cohomology sheaf $\mathcal{H}^i \mathcal{P} = \mathcal{H}^i K \oplus \mathcal{H}^i K'$. Therefore $\text{Supp } \mathcal{H}^i K \subset \text{Supp } \mathcal{H}^i \mathcal{P}$, and thus $\dim \text{Supp } \mathcal{H}^i K \leq \dim \text{Supp } \mathcal{H}^i \mathcal{P} \leq -i$. This proves the support condition. Note that $\mathcal{P}^\vee = K^\vee \oplus (K')^\vee$, the cosupport condition follows similarly. \square

5. PERVERSE FILTRATION OF SYMMETRIC PRODUCTS

In this section, we show that the perverse filtration of symmetric product is compatible with the perverse filtration of cartesian product of maps. We also use the symmetric product and the alternating product for perverse sheaves to give a perverse decomposition of symmetric product of maps.

Lemma 5.1. *Let X be a smooth quasi projective variety. Let $q : X^n \rightarrow X^{(n)}$. Let $K_i \in D_c^b(X)$, $i = 1, \dots, n$. Then \mathfrak{S}_n acts on*

$$\tilde{K} = \bigoplus_{\sigma \in \mathfrak{S}_n} K_{\sigma(1)} \boxtimes \cdots \boxtimes K_{\sigma(n)}$$

as an endomorphism. Furthermore, $(Rp_ \tilde{K})^{\mathfrak{S}_n} \cong Rp_*(K_1 \boxtimes \cdots \boxtimes K_n)$. More generally, let $\mathbf{k} = (k_1, \dots, k_m)$ with $k_1 + \cdots + k_m = n$. Let $\mathfrak{S}_{\mathbf{k}} = \{\sigma : [n] \rightarrow [m] : |f^{-1}(i)| = k_i\}$. Let $q_{\mathbf{k}} : X^{(k_1)} \times \cdots \times X^{(k_m)} \rightarrow X^{(n)}$. Then $\mathfrak{S}_{\mathbf{k}}$ acts on*

$$\tilde{K}_{\mathbf{k}} := \bigoplus_{\sigma \in \mathfrak{S}_{\mathbf{k}}} K_{\sigma(1)} \boxtimes \cdots \boxtimes K_{\sigma(n)},$$

and we have

$$(Rq_* \tilde{K}_{\mathbf{k}})^{\mathfrak{S}_n} \cong Rq_{\mathbf{k},*}(K_1^{(k_1)} \boxtimes \cdots \boxtimes K_m^{(k_m)}).$$

Similar result holds for alternating part

$$(Rq_* \tilde{K}_{\mathbf{k}})^{\text{sign}-\mathfrak{S}_n} \cong Rq_{\mathbf{k},*}(K_1^{\{k_1\}} \boxtimes \cdots \boxtimes K_m^{\{k_m\}}).$$

Proof. Note that \mathfrak{S}_n acts on \tilde{K} by permuting the direct summands (up to sign). The invariant part of the push-forward is determined by any one of its summands. \square

Proposition 5.2. *Let X be a smooth quasi-projective variety. Let $q : X^n \rightarrow X^{(n)}$. Let $K = \bigoplus_{i=1}^m K_i \in D_c^b(X)$. Then we have the expansion*

$$K^{(n)} \cong \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*}(K_1^{(k_1)} \boxtimes \cdots \boxtimes K_n^{(k_n)}).$$

and

$$K^{\{n\}} \cong \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*}(K_1^{\{k_1\}} \boxtimes \dots \boxtimes K_n^{\{k_n\}}).$$

Proof. By lemma 5.1, we have

$$\begin{aligned} K^{(n)} &= (Rq_* K^{\boxtimes n})^{\mathfrak{S}_n} \\ &= \left(Rq_* \bigoplus_{1 \leq i_1, \dots, i_n \leq m} K_{i_1} \boxtimes \dots \boxtimes K_{i_n} \right)^{\mathfrak{S}_n} \\ &= \left(Rq_* \bigoplus_{\mathbf{k}} \tilde{K}_{\mathbf{k}} \right)^{\mathfrak{S}_n} \\ &= \bigoplus_{\mathbf{k}} \left(Rq_* \tilde{K}_{\mathbf{k}} \right)^{\mathfrak{S}_n} \\ &= \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*} \left(K_1^{(k_1)} \boxtimes \dots \boxtimes K_m^{(k_m)} \right) \end{aligned}$$

□

Lemma 5.3. *Let $f : X \rightarrow Y$ be a proper morphism between smooth quasi-projective varieties. Let $K \in D_c^b(X)$. We have the following diagram*

$$\begin{array}{ccc} X^n & \xrightarrow{q} & X^{(n)} \\ \downarrow f^n & & \downarrow f^{(n)} \\ Y^n & \xrightarrow{q} & Y^{(n)} \end{array}$$

Then $Rf_*^{(n)} K^{(n)} \cong (Rf_* K)^{(n)}$.

Proof. $(Rf_* K)^{(n)} \cong (Rq_*(Rf_* K)^{\boxtimes n})^{\mathfrak{S}_n} \cong (Rq_* Rf_* K^{\boxtimes n})^{\mathfrak{S}_n} \cong \left(Rf_*^{(n)} Rq_* K^{\boxtimes n} \right)^{\mathfrak{S}_n} \cong Rf_*^{(n)} (Rq_* K^{\boxtimes n})^{\mathfrak{S}_n} \cong Rf_*^{(n)} K^{(n)}$. □

Proposition 5.4 (decomposition for symmetric products). *Let $f : X \rightarrow Y$ be a proper morphism between smooth quasi-projective varieties. Let*

$$Rf_* \mathbb{Q}_X[\dim X - r(f)] \cong \bigoplus_{i=0}^{2r(f)} \mathcal{P}_i[-i]$$

be the perverse decomposition, where \mathcal{P}_i are perverse sheaves on Y . Then the perverse decomposition of the map $f^{(n)} : X^{(n)} \rightarrow Y^{(n)}$ is given as follows. The formula is slightly different depending on the parity of $\dim X - r(f)$. When $\dim X - r(f)$ is even, then

$$\begin{aligned} & Rf_*^{(n)} \mathbb{Q}_{X^{(n)}}[n(\dim X - r(f))] \\ & \cong \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*} \left(\mathcal{P}_0^{(k_0)} \boxtimes (\mathcal{P}_1[-1])^{(k_1)} \boxtimes \dots \boxtimes (\mathcal{P}_{2r(f)}[-2r(f)])^{(k_{2r(f)})} \right) \\ & \cong \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*} \left(\mathcal{P}_0^{(k_0)} \boxtimes \mathcal{P}_1^{\{k_1\}} \boxtimes \dots \boxtimes \mathcal{P}_{2r(f)}^{(k_{2r(f)})} \right) \left[- \sum_{i=0}^{2r(f)} i k_i \right] \end{aligned}$$

where the all factors in the second line are symmetric products, while in the third line, factors are alternating products for odd subscripts and are symmetric products

for even subscripts. When $\dim X - r(f)$ is odd, all the symmetric product and alternating product are swapped.

Proof. By the well-known fact $(\mathbb{Q}_X)^{(n)} = \mathbb{Q}_{X^{(n)}}$, remark 4.4 and lemma 5.3, we have

$$Rf_*^{(n)} \mathbb{Q}_{X^{(n)}}[n(\dim X - r(f))] = \begin{cases} (Rf_* \mathbb{Q}_X[\dim X - r(f)])^{(n)} & \text{if } \dim X - r(f) \text{ is even.} \\ (Rf_* \mathbb{Q}_X[\dim X - r(f)])^{\{n\}} & \text{if } \dim X - r(f) \text{ is odd.} \end{cases}$$

Then we use proposition 5.2 to obtain the isomorphism. Using proposition 3.1, proposition 4.5 and the fact that the projection q_k is small, we know this isomorphism is indeed a perverse decomposition. \square

Although the perverse decomposition of symmetric product is somewhat complicated, the perverse filtration is much simpler. It is compatible with the one for the cartesian product as one may expected.

Lemma 5.5. *Let $f : X \rightarrow Y$ as before. Then*

$${}^p\tau_{\leq p} \left((Rf_* \mathbb{Q}_X)^{(n)} \right) = \left(Rq_* \left({}^p\tau_{\leq p} (Rf_* \mathbb{Q}_X)^{\boxtimes n} \right) \right)^{\mathfrak{S}_n}$$

Proof. Note that \mathfrak{S}_n invariant part is a direct summand, so it commutes with the functor ${}^p\tau_{\leq p}$. Furthermore, the quotient map q is small, hence Rq_* is t -exact. \square

Proposition 5.6. *Under the isomorphism*

$$H^* \left(X^{(n)} \right) = (H^*(X^n))^{\mathfrak{S}_n},$$

the perverse filtration can be identified as

$$P_p H^* \left(X^{(n)} \right) = (P_p H^*(X^n))^{\mathfrak{S}_n},$$

where the perversity in the parenthesis is taken with respect to $f^n : X^n \rightarrow Y^n$.

Proof. By lemma 5.3 and lemma 5.5, we have

$$\begin{aligned} {}^p\tau_{\leq p} Rf_*^{(n)} \mathbb{Q}_{X^{(n)}} &= {}^p\tau_{\leq p} (Rf_* \mathbb{Q}_X)^{(n)} \\ &= \left(Rq_* {}^p\tau_{\leq p} (Rf_* \mathbb{Q}_X)^{\boxtimes n} \right)^{\mathfrak{S}_n} \end{aligned}$$

After taking the cohomology, we have

$$\begin{aligned} P_p H^*(X^{(n)}) &= \mathbb{H} \left(Y^{(n)}, {}^p\tau_{\leq p} Rf_*^{(n)} \mathbb{Q}_{X^{(n)}} \right) \\ &= (P_p H^*(X^n))^{\mathfrak{S}_n} \end{aligned}$$

So the result follows. \square

For later use, we use the following notations, which is a direct generalization of the above. Partition of n are denoted as $\nu = 1^{a_1} \dots n^{a_n}$, where $\sum_{i=1}^n i a_i = n$. The length of partition is denoted as $l(\nu) = \sum_{i=1}^n a_i$. Let \mathfrak{S}_n be symmetric group of n elements. Denote $\mathfrak{S}_\nu = \mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_n}$. For a quasi-projective variety X , denote $X^{(\nu)} = X^{l(\nu)} / \mathfrak{S}_\nu = X^{(a_1)} \times \dots \times X^{(a_n)}$. For $K \in D_c^b(X)$, denote the multi-symmetric and multi-alternating external product by

$$K^{(\nu)} = \boxtimes_{i=1}^n K^{(a_i)} \in D_c^b(X^{(\nu)})$$

$$K^{\{\nu\}} = \boxtimes_{i=1}^n K^{\{a_i\}} \in D_c^b(X^{(\nu)})$$

We still have

$$(K[a])^{(\nu)} = \begin{cases} K^{(\nu)}[l(\nu)a] & a \text{ is even,} \\ K^{\{\nu\}}[l(\nu)a] & a \text{ is odd.} \end{cases}$$

In fact, since the external product is compatible with push-forward and perversity, all result in this section can be generalized to the multi-symmetric or multi-alternating context.

6. PERVERSE FILTRATION OF HILBERT SCHEMES OF FIBERED SURFACES

In this section, we give a perverse decomposition of a map from Hilbert scheme of a surface to the symmetric product of a curve. We compute the perverse filtration on the cohomology of the Hilbert scheme along the way. Let $f : S \rightarrow C$, be a proper map from a smooth quasi-projective surface to a smooth quasi-projective curve. Let $S^{[n]}$ denote the Hilbert scheme of n points on surface S . Then we have the following diagram.

$$\begin{array}{ccccc} & & & & S^{[n]} \\ & & & & \downarrow \pi \\ S^{l(\nu)} & \xrightarrow{/\mathfrak{S}_\nu} & S^{(\nu)} & \xrightarrow{r_S^{(\nu)}} & S^{(n)} \\ f^{l(\nu)} \downarrow & & f^{(\nu)} \downarrow & & \downarrow f^{(n)} \\ C^{l(\nu)} & \xrightarrow{/\mathfrak{S}_\nu} & C^{(\nu)} & \xrightarrow{r_C^{(\nu)}} & C^{(n)} \end{array} \quad \Bigg) \quad h$$

To obtain a perverse decomposition for mpa h , we need following result:

Theorem 6.1 ([2]). *Let S be an quasi-projective algebraic surface. Then we have the decomposition theorem for the Hilbert-Chow morphism.*

$$R\pi_* \mathbb{Q}_{S^{[n]}}[2n] \cong \bigoplus_{\nu} Rr_{S,*}^{(\nu)} \mathbb{Q}_{S^{(\nu)}}[2l(\nu)]$$

Proposition 6.2. *Let $f : S \rightarrow C$ be a proper map from smooth algebraic surface to smooth algebraic curve. Let*

$$Rf_* \mathbb{Q}_S[1] = \mathcal{P}_0 \oplus \mathcal{P}_1[-1] \oplus \mathcal{P}_2[-2]$$

be a perverse decomposition, where $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ are perverse sheaves on C . Then a perverse decomposition of the morphism $h : S^{[n]} \rightarrow C^{(n)}$ is given by:

$$\begin{aligned} Rh_* \mathbb{Q}_{S^{[n]}}[n] &\cong \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} Rr_{C,*}^{(\nu)} \left((\mathcal{P}_0[0] \oplus \mathcal{P}_1[-1] \oplus \mathcal{P}_2[-2])^{\{\nu\}} \right) [l(\nu) - n] \\ &\cong \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} Rr_{C,*}^{(\nu)} \left(\bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} Rq_{\mathbf{r},\mathbf{s},\mathbf{t},*} \mathcal{P}_0^{\{\mathbf{r}\}} \boxtimes \mathcal{P}_1^{(\mathbf{s})} \boxtimes \mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})] \end{aligned}$$

where $C(\mathbf{r}, \mathbf{s}, \mathbf{t}) = n - l(\nu) + \sum_{i=1}^n (s_i + 2t_i)$. $\mathbf{r} = (r_1, \dots, r_n)$, similar for \mathbf{s} and \mathbf{t} . $\mathbf{r} + \mathbf{s} + \mathbf{t} = \mathbf{a}$ means that $r_i + s_i + t_i = a_i$ holds for any i , and

$$q_{\mathbf{r},\mathbf{s},\mathbf{t}} : \prod_{i=1}^{l(\nu)} C^{(r_i)} \times C^{(s_i)} \times C^{(t_i)} \rightarrow \prod_{i=1}^{l(\nu)} C^{(a_i)}$$

Proof. The proof is formal.

$$\begin{aligned}
Rh_* \mathbb{Q}_{S^{[n]}}[n] &\cong Rf_*^{(n)} R\pi_* \mathbb{Q}_{S^{[n]}}[n] \\
&\cong Rf_*^{(n)} \left(\bigoplus_{\nu} Rr_{S,*}^{(\nu)} \mathbb{Q}_{S^{(\nu)}}[2l(\nu) - n] \right) \\
&\cong \bigoplus_{\nu} Rf_*^{(n)} Rr_{S,*}^{(\nu)} \mathbb{Q}_{S^{(\nu)}}[2l(\nu) - n] \\
&\cong \bigoplus_{\nu} Rr_{C,*}^{(\nu)} Rf_*^{(\nu)} \mathbb{Q}_{S^{(\nu)}}[2l(\nu) - n] \\
&\cong \bigoplus_{\nu} Rr_{C,*}^{(\nu)} (Rf_* \mathbb{Q}_S)^{(\nu)} [2l(\nu) - n] \\
&\cong \bigoplus_{\nu} Rr_{C,*}^{(\nu)} (Rf_* \mathbb{Q}_S[1])^{\{\nu\}} [l(\nu) - n] \\
&\cong \bigoplus_{\nu} Rr_{C,*}^{(\nu)} (\mathcal{P}_0[0] \oplus \mathcal{P}_1[-1] \oplus \mathcal{P}_2[-2])^{\{\nu\}} [l(\nu) - n] \\
&\cong \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} Rr_{C,*}^{(\nu)} \left(\bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} Rq_{\mathbf{r},\mathbf{s},\mathbf{t},*} \mathcal{P}_0^{\{\mathbf{r}\}} \boxtimes \mathcal{P}_1^{(\mathbf{s})} \boxtimes \mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})]
\end{aligned}$$

Here we use lemma 5.3 and proposition 5.4. Note that here the cohomological shift is odd, the symmetric product and alternating product is swapped. By proposition 3.1 and proposition 4.5 and the fact that $q_{\mathbf{r},\mathbf{s},\mathbf{t}}$ is finite, all terms in the parenthesis are perverse. Note that $r_{C,*}^{(\nu)}$ is a closed embedding, so $Rr_{C,*}^{(\nu)}$ is t -exact, it preserves perversity. Therefore the above gives a perverse decomposition. \square

Remark 6.3. In the proposition 6.2, we have symmetric product for odd perversity term and alternating products for even perversity terms. This counter-intuitive result is due to the fact that $\mathcal{P}_i[-i]$ are direct summands of $Rf_* \mathbb{Q}_S[1]$ rather than $Rf_* \mathbb{Q}_S$.

Corollary 6.4. *Under the isomorphism*

$$H^*(S^{[n]}) = \bigoplus_{\nu} \left(H^*(S^{l(\nu)}) \right)^{\mathfrak{S}_{\nu}} [2l(\nu) - 2n],$$

the perverse filtration can be identified as

$$P_p H^*(S^{[n]}) = \bigoplus_{\nu} \left(P_{p+l(\nu)-n} H^*(S^{l(\nu)}) \right)^{\mathfrak{S}_{\nu}} [2l(\nu) - 2n],$$

where the perversity in the parenthesis is taken with respect to $f^{l(\nu)} : S^{l(\nu)} \rightarrow C^{l(\nu)}$.

Proof. By t -exactness of $Rr_{C,*}^{(\nu)}$ and proposition 6.2, we have

$$\begin{aligned}
&{}^p\tau_{\leq p} Rh_* \mathbb{Q}_{S^{[n]}}[n] \\
&= \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} Rr_{C,*}^{(\nu)} \left({}^p\tau_{\leq p+l(\nu)-n} (Rf_* \mathbb{Q}_S[1])^{\{\nu\}} \right) [l(\nu) - n] \\
&= \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} Rr_{C,*}^{(\nu)} \left(Rq_* {}^p\tau_{\leq p+l(\nu)-n} (Rf_* \mathbb{Q}_S[1])^{\boxtimes l(\nu)} \right)^{\text{sign-}\mathfrak{S}_n} [l(\nu) - n]
\end{aligned}$$

where q is the quotient map denoted as $/\mathfrak{S}_\nu$ in the previous diagram and the last isomorphism is due to corollary 5.5. After taking the cohomology, we have

$$\begin{aligned}
& P_p H^*(S^{[n]})[n] \\
= & \mathbb{H}(C^{(n)}, {}^p\tau_{\leq p} Rh_* \mathbb{Q}_{S^{[n]}}[n]) \\
= & \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \mathbb{H}\left(C^{(\nu)}, \left(Rq_* {}^p\tau_{\leq p+l(\nu)-n}(Rf_* \mathbb{Q}_S[1])^{\boxtimes l(\nu)}\right)^{\text{sign}-\mathfrak{S}_n}\right) [l(\nu) - n] \\
= & \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \mathbb{H}\left(C^{l(\nu)}, {}^p\tau_{\leq p+l(\nu)-n}(Rf_* \mathbb{Q}_S[1])^{\boxtimes l(\nu)}\right)^{\text{sign}-\mathfrak{S}_n} [l(\nu) - n] \\
= & \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \left(P_{p+l(\nu)-n} H^*\left(C^{l(\nu)}, (Rf_* \mathbb{Q}_S[1])^{\boxtimes l(\nu)}\right)\right)^{\text{sign}-\mathfrak{S}_n} [l(\nu) - n] \\
= & \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \left(P_{p+l(\nu)-n} H^*\left(S^{l(\nu)}, (\mathbb{Q}_S[1])^{\boxtimes l(\nu)}\right)\right)^{\text{sign}-\mathfrak{S}_n} [l(\nu) - n] \\
= & \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \left(P_{p+l(\nu)-n} H^*\left(S^{l(\nu)}, \mathbb{Q}_S^{\boxtimes l(\nu)}\right)\right)^{\mathfrak{S}_n} [2l(\nu) - n] \\
= & \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \left(P_{p+l(\nu)-n} H^*(S^{l(\nu)})\right)^{\mathfrak{S}_n} [2l(\nu) - n]
\end{aligned}$$

So the result follows. \square

We may also define function \mathfrak{p} on cohomology classes in $H^*(S^{[n]})$, but it will be much easier if we use the notation introduced by Lehn in [13]. Therefore we postpone the definition to section 8.

7. CUP PRODUCT OF HILBERT SCHEME OF POINTS FOR K3 SURFACES

In this section we recall the notation, definition and results on the cup product of Hilbert scheme of points for K3 surfaces. All results in this section are due to Lehn's paper [13].

Let S be a projective surface with numerically trivial canonical bundle. Let $A = H^*(S; \mathbb{Q})$ be the cohomology with \mathbb{Q} coefficient. Let $[n]$ denote the set $\{1, \dots, n\}$.

Definition 7.1. Let I be a finite set of cardinality n . Define

$$A^I = \left(\bigoplus_{f: [n] \xrightarrow{\sim} I} A_{f(1)} \otimes \dots \otimes A_{f(n)} \right) / \mathfrak{S}_n$$

Remark 7.2. In fact, A^I is isomorphic to $A^{|I|}$. This isomorphism is canonical once an isomorphism $\varphi: [n] \rightarrow I$ is fixed.

Definition 7.3. Let $\varphi: I \rightarrow J$ be a surjective map between sets. Then φ induces a morphism $\varphi: S^J \rightarrow S^I$ by sending $(x_1, \dots, x_{|J|})$ to $(x_{\varphi(1)}, \dots, x_{\varphi(|I|)})$. Define φ_* and φ^* to be the push-forward and pull-back map in cohomology associated to φ .

Remark 7.4. The pull-back map can be described explicitly as follows. First note that φ is a product of diagonal embedding map: the j -th copy of S in S^J is embedded diagonally in $S^{f^{-1}(j)}$. Pulling-back along diagonal embedding is exactly

the definition of cup product. So if we fix isomorphism $f : [n] \xrightarrow{\sim} I$, $g : [m] \xrightarrow{\sim} J$, we will have

$$\tilde{\varphi} : [n] \rightarrow I \rightarrow J \rightarrow [m]$$

Therefore

$$\begin{aligned} \tilde{\varphi}^* : A^n &\rightarrow A^m \\ a_1 \otimes \cdots \otimes a_n &\mapsto \bigotimes_{j=1}^m \bigcup_{i \in \tilde{\varphi}^{-1}(j)} a_i \end{aligned}$$

and

$$\varphi^* : A^I \rightarrow A^n \xrightarrow{\tilde{\varphi}^*} A^m \rightarrow A^J$$

It is easy to check that this is independent of choice of f and g .

Now we define a wreath product of A and \mathfrak{S}_n , which is used to describe the cohomology of Hilbert scheme of points. For a permutation $\sigma \in \mathfrak{S}_n$ and a partition $\nu = 1^{a_1} \cdots n^{a_n}$ of n , we say σ is of type ν if σ has exactly a_i i -cycles. For K a subgroup of \mathfrak{S}_n , and K -stable subset $E \subset [n]$, let $K \backslash E$ denote the set of orbits for the induced action of K on E .

Definition 7.5. For $\sigma, \tau \in \mathfrak{S}_n$, the graph defect $g(\sigma, \tau) : \langle \sigma, \tau \rangle \backslash [n] \rightarrow \mathbb{Q}$ is defined by

$$g(\sigma, \tau)(E) = \frac{1}{2}(|E| + 2 - |\langle \sigma \rangle \backslash E| - |\langle \tau \rangle \backslash E| - |\langle \sigma\tau \rangle \backslash E|).$$

In fact, $g(\sigma, \tau)$ is always non-negative integer.

Definition 7.6.

$$A\{\mathfrak{S}_n\} := \bigoplus_{\sigma \in \mathfrak{S}_n} A^{\otimes \langle \sigma \rangle \backslash [n]}[-2|\sigma|] \cdot \sigma$$

\mathfrak{S}_n acts on $A\{\mathfrak{S}_n\}$ as follows: the action of $\tau \in \mathfrak{S}_n$ on $[n]$ induces a bijection

$$\begin{aligned} \sigma : \langle \sigma \rangle \backslash [n] &\rightarrow \langle \tau\sigma\tau^{-1} \rangle \backslash [n] \\ x &\mapsto \tau x \end{aligned}$$

for each σ and hence an isomorphism

$$\begin{aligned} \tilde{\tau} : A\{\mathfrak{S}_n\} &\rightarrow A\{\mathfrak{S}_n\} \\ a\sigma &\mapsto \tau^*(a)\tau\sigma\tau^{-1} \end{aligned}$$

Let

$$A^{[n]} := (A\{\mathfrak{S}_n\})^{\mathfrak{S}_n}$$

be the subspace of invariants.

Any inclusion $H \subset K$ of subgroups of \mathfrak{S}_n induces a surjection $H \backslash [n] \twoheadrightarrow K \backslash [n]$ of set of orbits and hence induces pull-back map

$$f^{H,K} : A^{\otimes H \backslash [n]} \rightarrow A^{\otimes K \backslash [n]}$$

and push-forward map

$$f_{K,H} : A^{\otimes K \backslash [n]} \rightarrow A^{\otimes H \backslash [n]}$$

Definition 7.7. For $\sigma, \tau \in \mathfrak{S}_n$, define

$$\begin{aligned} m_{\sigma,\tau} : A^{\otimes \langle \sigma \rangle \backslash [n]} \otimes A^{\otimes \langle \tau \rangle \backslash [n]} &\rightarrow A^{\otimes \langle \sigma\tau \rangle \backslash [n]} \\ a \otimes b &\mapsto f_{\langle \sigma,\tau \rangle, \langle \sigma\tau \rangle}(f^{\langle \sigma \rangle, \langle \sigma,\tau \rangle}(a) \cdot f^{\langle \tau \rangle, \langle \sigma,\tau \rangle}(b) \cdot e^{g(\sigma,\tau)}) \end{aligned}$$

where e is the Euler class of S .

Proposition 7.8. *The product $A\{\mathfrak{S}_n\} \times A\{\mathfrak{S}_n\} \rightarrow A\{\mathfrak{S}_n\}$ defined by*

$$a\sigma \cdot b\tau := m_{\sigma,\tau}(a \otimes b)\sigma\tau$$

is associative and \mathfrak{S}_n -equivariant. So it descends to a product on $A^{[n]}$.

Theorem 7.9. *Let S be a smooth projective surface with numerically trivial canonical divisor. Then there is a canonical isomorphism of graded rings*

$$H^*(S; \mathbb{Q})^{[n]} \xrightarrow{\cong} H^*(S^{[n]}; \mathbb{Q})$$

8. MULTIPLICATIVITY OF PERVERSE FILTRATION

Definition 8.1. Let $f : S \rightarrow C$ be a morphism from smooth algebraic surface to smooth algebraic curve. Let B be the basis obtained in proposition 2.4. Then we have

$$B\{\mathfrak{S}_n\} := \left\{ \bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \cdot \sigma \mid \alpha_{ij} \in B, \sigma \text{ is of type } 1^{a_1} \cdots n^{a_n} \right\}$$

is a basis of $H^*(X)\{\mathfrak{S}_n\}$. We define abstract perversity on $B\{\mathfrak{S}_n\}$ as

$$\mathfrak{p} \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \cdot \sigma \right) = \sum_{i=1}^n \sum_{j=1}^{a_i} \mathfrak{p}(\alpha_{ij}) + \sum_{i=1}^n (i-1)a_i,$$

and extend by linearity in the sense that

$$P_p H^*(S)\{\mathfrak{S}_n\} = \text{Span}\{\beta \in B\{\mathfrak{S}_n\} \mid \mathfrak{p}(\beta) \leq p\}.$$

In particular, the basis is filtered with respect to the abstract perverse filtration.

Proposition 8.2. *The abstract perversity is invariant under \mathfrak{S}_n -action. Furthermore, after restricted to $H^*(S^{[n]})$, it is the same as the perversity given by the morphism $h : S^{[n]} \rightarrow C^{(n)}$ in corollary 6.4.*

Proof. Note that \mathfrak{S}_n acts on cohomology by permuting the factors, so the abstract perversity is invariant under S_n action. By definition, we have

$$n - l(\nu) = \sum_{i=1}^n (i-1)a_i$$

and

$$\mathfrak{p} \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \cdot \sigma \right) = p \text{ if and only if } \mathfrak{p} \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \right) = p + n - l(\nu).$$

where $\nu = 1^{a_1} \cdots n^{a_n}$ be the partition of n . Comparing with corollary 6.4, the abstract perversity is the same as geometric perversity induced by morphism h on the basis. Note that on both hand side the basis are filtered with respect to perverse filtration, so the abstract perverse filtration coincides with geometric perverse filtration. \square

Proposition 8.3. *Let $f : S \rightarrow C$ be a proper surjective morphism from smooth algebraic surface to smooth algebraic curve. Then the perverse filtration on $H^*(S)$ is multiplicative.*

Proof. The perverse decomposition of a proper map from surface to curve is due to theorem 3.2.3 of [4]. Let $f : S \rightarrow C$ be a proper surjective map from a smooth surface to a smooth curve. Let $\hat{f} : \hat{S} \rightarrow \hat{C}$ be its smooth part. Let $j : \hat{C} \rightarrow C$ be the open embedding. Let $\hat{R}^i = R^i \hat{f}_* \mathbb{Q}_{\hat{S}}$. Then one has a non-canonical perverse decomposition of map f :

$$Rf_* \mathbb{Q}_S[1] \cong \left\{ j_* \hat{R}^0[1] \right\} \bigoplus \left\{ j_* \hat{R}^1[1] \oplus \bigoplus_{p \in C \setminus \hat{C}} \mathbb{Q}_p^{n_p-1} \right\} [-1] \bigoplus \{ j_* \hat{R}^2[1] \} [-2]$$

where n_p is the number of irreducible components of fiber over p . Note that the ordinary Leray filtration is multiplicative, and the only difference between ordinary Leray filtration and the perverse filtration is that the classes corresponding to $\bigoplus_{p \in C \setminus \hat{C}} \mathbb{Q}_p^{n_p-1}$ is shifted from $R^2 f_* \mathbb{Q}_S[-2]$ to $\mathcal{P}_1[-1]$. Therefore, it suffices to check cup products with these classes. Furthermore, these classes are in perversity 1, the only possibility to violate the multiplicativity is that their cup with perversity 0 classes have perversity 2. However, they are fundamental classes of irreducible components of special fibers, and perversity 0 classes are pull-backs of classes on the curve. They don't meet if the pull-back class is not the surface itself, and cupping with fundamental class of the surface is identity map. So in both cases, the multiplicativity is preserved. \square

Theorem 8.4. *Let $f : S \rightarrow C$ be a surjective morphism from a smooth projective surface with numerically trivial canonical bundle to a smooth projective curve. Then the perverse filtration of $H^*(S^{[n]}; \mathbb{Q})$ with respect to the morphism $h : S^{[n]} \rightarrow C^{(n)}$ is multiplicative, namely, we have*

$$P_p H^*(S^{[n]}; \mathbb{Q}) \cup P_{p'} H^*(S^{[n]}; \mathbb{Q}) \subset P_{p+p'} H^*(S^{[n]}; \mathbb{Q})$$

To prove this theorem, we need two lemmata,

Lemma 8.5. *For surjective map between sets $\varphi : I \twoheadrightarrow J$, the pullback map $\varphi^* : H^*(S)^I \rightarrow H^*(S)^J$ doesn't increase perversity.*

Proof. By corollary 3.2, the basis $B^{|I|}$ is filtered, so it suffices to check for the basis. Pick element $\alpha_1 \otimes \cdots \otimes \alpha_{|I|} \in B^{|n|}$, then

$$\varphi^*(\alpha_1 \otimes \cdots \otimes \alpha_{|I|}) = \bigotimes_{j=1}^{|J|} \bigcup_{i \in \varphi^{-1}(j)} \alpha_i$$

Note that

$$\mathbf{p}(\alpha_1 \otimes \cdots \otimes \alpha_{|I|}) = \sum_{j=1}^{|I|} \alpha_j$$

and

$$\mathbf{p} \left(\bigotimes_{j=1}^{|J|} \bigcup_{i \in \varphi^{-1}(j)} \alpha_i \right) \leq \sum_{j=1}^{|J|} \sum_{i \in \varphi^{-1}(j)} \mathbf{p}(\alpha_i) = \sum_{j=1}^{|I|} \alpha_j$$

The inequality is due to proposition 3.1 and lemma 8.3. So the result follows. \square

Lemma 8.6. *For surjective map between $\varphi : I \twoheadrightarrow J$, the push-forward map*

$$\varphi_* : H^*(S)^J \rightarrow H^*(S)^I$$

increases the perversity at most by $2(|I| - |J|)$.

Proof. Again, it suffices to prove for basis $\alpha_1 \otimes \cdots \otimes \alpha_{|J|} \in B^{|J|}$. Let $b_j = |\varphi^{-1}(j)|$. By definition,

$$\varphi_*(\alpha_1 \otimes \cdots \otimes \alpha_{|J|}) = \pm \bigotimes_{j=1}^{|J|} \Delta_{b_j,*}(\alpha_j)$$

where the sign is to put the factors to the correct place. By proposition 3.3, we have

$$\begin{aligned} \mathfrak{p} \left(\pm \bigotimes_{j=1}^{|J|} \Delta_{b_j,*}(\alpha_j) \right) &\leq \sum_{j=1}^{|J|} \mathfrak{p}(\Delta_{b_j,*}(\alpha_j)) \\ &= \sum_{j=1}^{|J|} \mathfrak{p}(\alpha_j) + 2(b_j - 1) \\ &= \sum_{j=1}^{|J|} \mathfrak{p}(\alpha_j) + 2 \sum_{j=1}^{|J|} b_j - 2 \sum_{j=1}^{|J|} 1 \\ &= \sum_{j=1}^{|J|} \mathfrak{p}(\alpha_j) + 2(|I| - |J|) \end{aligned}$$

□

Proof of Theorem 8.4. It suffices to prove the abstract perverse filtration defined on $H^*(S; \mathbb{Q})\{\mathfrak{S}_n\}$ is multiplicative. Furthermore, it suffices to prove the result for basis, namely

$$\begin{aligned} &\mathfrak{p} \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \cdot \sigma \cup \bigotimes_{i=1}^n \bigotimes_{j=1}^{a'_i} \alpha'_{ij} \cdot \tau \right) \\ &\leq \sum_{i=1}^n \sum_{j=1}^{a_i} \mathfrak{p}(\alpha_{ij}) + \sum_{i=1}^n (i-1)a_i \\ &+ \sum_{i=1}^n \sum_{j=1}^{a'_i} \mathfrak{p}(\alpha'_{ij}) + \sum_{i=1}^n (i-1)a'_i, \end{aligned}$$

where α_{ij} and α'_{ij} run over basis B obtained in proposition 2.4. Note that the cup product formula actually works on each orbit of $\langle \sigma, \tau \rangle$ -action on $[n]$ individually. Let E be an orbit of the action $\langle \sigma, \tau \rangle$ on $[n]$, i.e. $|\langle \sigma, \tau \rangle \backslash E| = 1$. The product is computed by

$$\begin{aligned} A^{\otimes \langle \sigma \rangle \backslash E} \cdot \sigma|_E \otimes A^{\otimes \langle \tau \rangle \backslash E} \cdot \tau|_E &\rightarrow A^{\otimes \langle \sigma \tau \rangle \backslash E} \cdot \sigma \tau|_E \\ a \cdot \sigma|_E \otimes a' \cdot \tau|_E &\mapsto f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle}(f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle}(a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle}(a') \cdot e^{g(\sigma, \tau)}) \cdot \sigma \tau|_E \end{aligned}$$

for every E . Note that Euler class e is of top degree, hence $e^g = 0$ for $g \geq 2$, it suffices to consider the following two cases.

(1) $g(\sigma, \tau) = 0$. By lemma 8.5 and lemma 8.6, we have

$$\begin{aligned}
& \mathbf{p} \left(f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle} (f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle} (a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle} (a') \cdot e^{g(\sigma, \tau)}) \cdot \sigma \tau|_E \right) \\
&= \mathbf{p} \left(f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle} (f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle} (a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle} (a')) \right) + |B| - |\langle \sigma \tau \rangle \setminus E| \\
&= \mathbf{p} \left(f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle} (a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle} (a') \right) + 2(|\langle \sigma \tau \rangle \setminus E| - 1) + |E| - |\langle \sigma \tau \rangle \setminus E| \\
&= \mathbf{p}(a) + \mathbf{p}(a') + |E| + |\langle \sigma \tau \rangle \setminus E| - 2 \\
&= \mathbf{p}(a \cdot \sigma) - (|E| - |\langle \sigma \rangle \setminus E|) + \mathbf{p}(a' \cdot \tau) - (|E| - |\langle \tau \rangle \setminus E|) + |E| + |\langle \sigma \tau \rangle \setminus E| - 2 \\
&= \mathbf{p}(a \cdot \sigma) + \mathbf{p}(a' \cdot \tau) - 2g(\sigma, \tau) \\
&= \mathbf{p}(a \cdot \sigma) + \mathbf{p}(a' \cdot \tau)
\end{aligned}$$

(2) $g(\sigma, \tau) = 1$. Since e itself is already at top degree, so only nonzero case is $a = 1$ and $a' = 1$. Then

$$\begin{aligned}
& \mathbf{p} \left(f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle} (f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle} (1) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle} (1) \cdot e^{g(\sigma, \tau)}) \cdot \sigma \tau|_E \right) \\
&= \mathbf{p} (f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle} (e)) + |E| - |\langle \sigma \tau \rangle \setminus E| \\
&= 2 + 2(|\langle \sigma \tau \rangle \setminus E| - 1) + |E| - |\langle \sigma \tau \rangle \setminus E| \\
&= |E| + |\langle \sigma \tau \rangle \setminus E| \\
&= |E| - |\langle \sigma \rangle \setminus E| + |E| - |\langle \tau \rangle \setminus E| \\
&= \mathbf{p}(1 \cdot \sigma) + \mathbf{p}(1 \cdot \tau)
\end{aligned}$$

The last but one equality is due to $g(\sigma, \tau) = 1$, which means $|E| = |\langle \sigma \rangle \setminus E| + |\langle \tau \rangle \setminus E| + |\langle \sigma \tau \rangle \setminus E|$.

□

By the theorem, we have the following result.

Theorem 8.7. *Let S be an elliptic K3 surface and $f : S \rightarrow \mathbb{P}^1$ be the elliptic fibration. Then the perverse filtration on $H^*(S^{[n]})$ defined by natural map $h : S^{[n]} \rightarrow \mathbb{P}^n$ is multiplicative.*

9. THE NON-COMPACT CASES

In this section we deal with five cases when S and C are non-compact, and show that perverse filtration of $h : S^{[n]} \rightarrow C^{(n)}$ is multiplicative. We first make the folklore definition for “our five cases”.

Definition 9.1. “Our five cases” refers to the following fibered surfaces.

- (1) \widetilde{A}_0 case. Let E be any elliptic curve. Then fixing a trivialization yields a natural map $f : T^*E \rightarrow \mathbb{C}$.
- (2) \widetilde{D}_4 case. Let E be any elliptic curve. The $\Gamma = \mathbb{Z}_2$ action on E induces an involution on T^*E . Let $S = \widetilde{T^*E/\mathbb{Z}_2}$ be the resolution of singularities. Let $C = \mathbb{C}/\mathbb{Z}_2 = \mathbb{C}$. Let $f : S \rightarrow T^*E/\mathbb{Z}_2 \rightarrow \mathbb{C}$.
- (3) \widetilde{E}_6 case. Let E be elliptic curve which admits $\Gamma = \mathbb{Z}_3$ action. The \mathbb{Z}_3 action on E induces an equivariant \mathbb{Z}_3 action on T^*E . Let $S = \widetilde{T^*E/\mathbb{Z}_3}$ be the resolution of singularities. Let $C = \mathbb{C}/\mathbb{Z}_3 = \mathbb{C}$. Let $f : S \rightarrow T^*E/\mathbb{Z}_3 \rightarrow \mathbb{C}$.

- (4) \widetilde{E}_7 case. Let E be any elliptic curve which admits $\Gamma = \mathbb{Z}_4$ action. The \mathbb{Z}_4 action on E induces an equivariant \mathbb{Z}_4 action on T^*E . Let $S = \widetilde{T^*E/\mathbb{Z}_4}$ be the resolution of singularities. Let $C = \mathbb{C}/\mathbb{Z}_4 = \mathbb{C}$. Let $f : S \rightarrow T^*E/\mathbb{Z}_4 \rightarrow \mathbb{C}$.
- (5) \widetilde{E}_8 case. Let E be any elliptic curve which admits $\Gamma = \mathbb{Z}_6$ action. The \mathbb{Z}_6 action on E induces an equivariant \mathbb{Z}_6 action on T^*E . Let $S = \widetilde{T^*E/\mathbb{Z}_6}$ be the resolution of singularities. Let $C = \mathbb{C}/\mathbb{Z}_6 = \mathbb{C}$. Let $f : S \rightarrow T^*E/\mathbb{Z}_6 \rightarrow \mathbb{C}$.

Proposition 9.2. *In our five cases, S has trivial canonical bundle. The dual graph of irreducible components of the fiber over 0 is affine Dynkin diagram $\widetilde{A}_0, \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, respectively. Let $\hat{f} : f^{-1}\mathbb{C}^* \rightarrow \mathbb{C}^*$ be the smooth part of the map, let $j : \mathbb{C}^* \rightarrow \mathbb{C}$, let $\hat{R}^1 = R^1\hat{f}_*\mathbb{Q}$. Then a perverse decomposition of $f : S \rightarrow \mathbb{C}$ can be written as follows.*

$$Rf_*\mathbb{Q}_S[1] \cong \{\mathbb{Q}_{\mathbb{C}}[1]\} \bigoplus \left\{ j_*\hat{R}^1 \oplus \mathbb{Q}_0^k \right\} [-1] \bigoplus \{\mathbb{Q}_{\mathbb{C}}[1]\} [-2]$$

where

$$k = \begin{cases} 0 & \widetilde{A}_0 \text{ case} \\ 4 & \widetilde{D}_4 \text{ case} \\ 6 & \widetilde{E}_6 \text{ case} \\ 7 & \widetilde{E}_7 \text{ case} \\ 8 & \widetilde{E}_8 \text{ case} \end{cases}$$

In particular, the dimension perverse filtration is given by

$$\dim \mathrm{Gr}_p H^d(X) = \begin{cases} 1 & p = d = 0 \\ 1 & p = d = 2 \\ 2 & p = d = 1, \widetilde{A}_0 \text{ case} \\ k & p = 1, d = 2, \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8 \text{ case} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The quotient of T^*E by Γ is computed using elementary methods. We list the type of singularities in our five cases.

Cases	Singularities
\widetilde{A}_0	none
\widetilde{D}_4	4 A_1
\widetilde{E}_6	3 A_2
\widetilde{E}_7	1 A_1 , 2 A_3
\widetilde{E}_8	1 A_1 , 1 A_2 , 1 A_5

Note that all singularities take place in the fiber over 0, so the dual graph of irreducible components match the affine Dynkin diagram. The action of Γ on T^*E preserves the canonical form, so the trivial canonical bundle descends to the quotient. Resolution of type A singularities is crepant, so S has trivial canonical bundle. The perverse decomposition is again due to theorem 3.2.2 of [4]. Here the map $f : S \rightarrow \mathbb{C}$ has connected fiber, so $j_*\hat{R}^0 = j_*\hat{R}^2 = \mathbb{Q}_{\mathbb{C}}$. The dimension of perverse filtration will follow if we show $\mathbb{H}^*(j_*\hat{R}^1) = 0$ in $\widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ cases. In

fact, \hat{R}^1 can be described explicitly. They are rank 2 representation of $\mathbb{Z} = \pi_1(C^*)$ with monodromy $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. It is easy to use Čech cohomology argument to show that all cohomology group of \hat{R}^1 vanishes, so does $j_*\hat{R}^1$. \square

When S is smooth and non-compact, the small diagonal embedding $\Delta_{n,*} : S \rightarrow S^n$ is proper. So we have push forward in Borel-Moore homology. We may still define

$$\Delta_{*,n} : H^*(S) \cong H_{4-*}^{BM}(S) \rightarrow H_{4-*}^{BM}(S^n) \cong H^{*+4(n-1)}(S^n)$$

Then following proposition is a counterpart of proposition 3.3.

Proposition 9.3. *Let $f : S \rightarrow \mathbb{C}$ be any of our five cases. In \widetilde{A}_0 case, the Gysin push-forward of small diagonal embedding $\Delta_{n,*}(\gamma) = 0$ for any $\gamma \in H^*(S)$ and $n > 1$. In other four cases, Let E_i be exceptional divisors of the resolution, then*

$$\Delta_{2,*}(1) = - \sum_{i=1}^k [E_i] \otimes [E_i]$$

and $\Delta_{n,*}(\gamma) = 0$ for any $n > 2$ or $n = 2, \gamma \neq 1$. In particular, the perversity estimation of diagonal in proposition 3.3

$$\mathbf{p}(\Delta_{n,*}(\gamma)) \leq \mathbf{p}(\gamma) + 2(n-1)$$

is still true.

Proof. Note that $\Delta_{n,*}$ increases the degree by $4(n-1)$, however in our cases, the top nontrivial degree for $H^*(S^n)$ is $2n$. So when $n \geq 3$, the push-forward is automatically 0. When $n = 2$, the only possible nonzero term is $\Delta_{2,*}(1)$. In \widetilde{A}_0 case, $H^4(S \times S)$ is one dimensional, generated by the class $[\mathbb{C}] \otimes [\mathbb{C}]$ and $H_4(S \times S)$ is generated by $E \otimes E$. $\langle \Delta_{2,*}(1), E \otimes E \rangle_{S \times S} = \langle E, E \rangle_S = 0$, so we have $\Delta_{2,*}(1) = 0$. For other four cases, according to the decomposition, we pick basis $[E_1], \dots, [E_k], \Sigma \in H^2(S)$, where Σ is a generic section of map $f : S \rightarrow \mathbb{C}$ whose perversity is 2. To write $\Delta_{2,*}(1)$ into the basis, it suffices to intersect it with the dual basis. The dual basis in $H_2(S)$ is E_1, \dots, E_k and F , where F denote the cycle class of generic fiber. Since

$$\langle \Delta_{2,*}(1), E_i \otimes E_j \rangle_{S \times S} = \langle E_i, E_j \rangle_S = -\delta_{ij}$$

$$\langle \Delta_{2,*}(1), E_i \otimes F \rangle_{S \times S} = \langle E_i, F \rangle_S = 0$$

$$\langle \Delta_{2,*}(1), F \otimes F \rangle_{S \times S} = \langle F, F \rangle_S = 0$$

We conclude that

$$\Delta_{2,*}(1) = - \sum_{i=1}^k [E_i] \otimes [E_i]$$

\square

Theorem 9.4. *In our five cases, the perverse filtration associated to map $h : S^{[n]} \rightarrow \mathbb{C}^n$ is multiplicative .*

Proof. First, Lehn's theorem on cup product of Hilbert schemes works for our five cases of non-compact surfaces. In all of our five cases, S can be compactified to \bar{S} such that the restriction map $H^*(\bar{S}) \rightarrow H^*(S)$ is surjective, so does $H^*(\bar{S}^{[n]}) \rightarrow H^*(S^{[n]})$. To compute cup product, we may pull-back to the compact cases, use the result for nontrivial canonical bundle case and then restrict. Since the extra terms involving canonical bundle vanish after restricting, the cup product formula still hold for our five cases.

The pull-back lemma 8.5 holds regardless of compact or not. The push-forward lemma 8.6 holds once we have perversity control of diagonal proposition 9.3. In non-compact case Euler class $e = 0$, so case (2) in theorem 8.4 is trivial. The only nontrivial case follows line by line as in the proof of case (1) in theorem 8.4. \square

In fact, the surfaces and their Hilbert schemes we consider are the moduli of parabolic Higgs bundles subject to certain numerical conditions. By the work of Gröchenig in [9], the Hilbert scheme is certain kind of moduli spaces of parabolic Higgs bundle, and the map h is the Hitchin map. To be precise, we consider the following moduli spaces

- (1) Degree 0 Higgs line bundle on elliptic curve E .
- (2) Let C denote \mathbb{P}^1 with 4 marked points. We consider moduli of degree 0 rank 2 parabolic Higgs bundle on C , whose Higgs field can have at worst order 1 pole at marked points and the residue of Higgs bundle is nilpotent with respect to a complete flag at each fiber of marked point.
- (3) Let C denote \mathbb{P}^1 with 3 marked points. We consider moduli of degree 0 rank 3 parabolic Higgs bundle on C , whose Higgs field can have at worst order 1 pole at marked points and the residue of Higgs bundle is nilpotent with respect to a complete flag at each fiber of marked point.
- (4) Let C denote \mathbb{P}^1 with 3 marked points. We consider moduli of degree 0 rank 4 parabolic Higgs bundle on C , whose Higgs field can have at worst order 1 pole at marked points. The residue of Higgs bundle is nilpotent with respect to a complete flag at fibers of two marked point, and a dimension $\{4, 2, 0\}$ -partial flag at the third.
- (5) Let C denote \mathbb{P}^1 with 3 marked points. We consider moduli of degree 0 rank 6 parabolic Higgs bundle on C , whose Higgs field can have at worst order 1 pole at marked points. The residue of Higgs bundle is nilpotent with respect to a complete flag at the first marked point, a dimension $\{6, 4, 2, 0\}$ -partial flag of the second and a dimension $\{6, 3, 0\}$ -partial flag of the third.

Theorem 9.5 ([9] Theorem 4.1). *We consider the moduli of parabolic Higgs bundle of any of the above five cases. Let $\Gamma = 0, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ in five cases, respectively. Then the moduli of parabolic Higgs bundles is naturally algebraically isomorphic Γ -equivariant Hilbert scheme on T^*E , which is the crepant resolution of quotient T^*E/Γ .*

Let (Q, λ) be a quiver with a marked vertex v . We denote by Q' the quiver obtained by adding an extra vertex v' which is only connected to v by an edge. We denote by λ' the dimension vector satisfying $\lambda'|_Q = \lambda$ and $\lambda'(v') = 1$.

Theorem 9.6 ([9] Theorem 5.1). *Let M denote the moduli space of parabolic Higgs bundle of any of the five type with the choice of parabolic weights as above. Then we have*

$$M^{[n]} \cong M_D(Q', (n\lambda)'),$$

where the parabolic weights at the marked point corresponding to $0 \in E$ are $\alpha_i = i/n$ for $i < n$ and $1 > \alpha_n > \frac{n-1}{n}$, and all the other weights are given by canonical weights; and the orbifold degree is 0. The Hitchin map $M^{[n]} \rightarrow \mathbb{A}^n$ factors through the Hilbert-Chow map

$$M^{[n]} \rightarrow M^{(n)} \rightarrow (\mathbb{A}^1)^{(n)} = \mathbb{A}^n,$$

where $M^{(n)} \rightarrow (\mathbb{A}^1)^{(n)}$ is induced by $M^n \rightarrow (\mathbb{A}^1)^n$.

Combine theorem 9.4 and theorem 9.6, we have

Theorem 9.7. *For five families of moduli space of parabolic Higgs bundles described in theorem 9.6, the perverse filtration associated to the Hitchin map is multiplicative.*

10. FULL VERSION OF $P = W$ FOR $n = 1$

In this section, we describe the corresponding character varieties for our five cases in definition 9.1, and we prove the full version of $P = W$ conjecture in these cases by using the explicit geometry of the surfaces.

By Simpson's table (page 720 of [15]), the moduli descriptions of the corresponding character varieties M_B in our five cases are:

- (1) Let E be any elliptic curve. Consider $GL(1, \mathbb{C})$ -representation of $\pi_1(E \setminus p)$ with no parabolic condition.
- (2) Let $C = \mathbb{P}^1 \setminus \{p_1, \dots, p_4\}$. Consider $GL(2, \mathbb{C})$ -representation of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalues are of type

$$(1, 1)(1, 1)(1, 1)(1, 1).$$

- (3) Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(3, \mathbb{C})$ -representation of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalues are of type

$$(1, 1, 1)(1, 1, 1)(1, 1, 1).$$

- (4) Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(4, \mathbb{C})$ -representation of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalues are of type

$$(2, 2)(1, 1, 1, 1)(1, 1, 1, 1).$$

- (5) Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(6, \mathbb{C})$ -representation of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalues are of type

$$(3, 3)(2, 2, 2)(1, 1, 1, 1, 1, 1).$$

We have explicit description for these character varieties.

Theorem 10.1 (Theorem 6.14 and 6.19 of [8]). *The character varieties M_B above can be described explicitly as follows.*

- (1) $\mathbb{C}^* \times \mathbb{C}^*$.
- (2) Degree 3 del Pezzo surface with a triangle removed.
- (3) Degree 3 del Pezzo surface with a nodal \mathbb{P}^1 removed.
- (4) Degree 2 del Pezzo surface with a nodal \mathbb{P}^1 removed.

(5) Degree 1 del Pezzo surface with a nodal \mathbb{P}^1 removed.

Furthermore, these del Pezzo surfaces can be expressed by explicit formula in weighted projective space away from the singularities, and the removed triangle or nodal \mathbb{P}^1 are cut out by a hyperplane section.

Lemma 10.2. *Let X be a del Pezzo surface in the theorem 10.1. Let $i : T \rightarrow X$ be an closed embedding of the removed curve in theorem 10.1 and $j : U \hookrightarrow X$ be its complement. Then*

$$W_2 H^2(U) \cong \text{Im} (H_c^2(U) \rightarrow H^2(U))$$

Proof. We have a diagram with the row and the column are distinguished triangles

$$\begin{array}{ccccc} & & i_* i^! \mathbb{Q}_X & & \\ & & \downarrow & & \\ Rj_! \mathbb{Q}_U & \longrightarrow & \mathbb{Q}_X & \longrightarrow & i_* \mathbb{Q}_T \\ & & \downarrow & & \\ & & Rj_* \mathbb{Q}_U & & \end{array}$$

Taking the cohomology at degree 2, we have

$$\begin{array}{ccccc} & & H_T^2(X) & & \\ & & \downarrow i_* & \searrow \psi & \\ H_c^2(U) & \xrightarrow{j_!} & H^2(X) & \xrightarrow{i^*} & H^2(T) \\ & \searrow \phi & \downarrow j^* & & \\ & & H^2(U) & & \end{array}$$

By [7], Corollaire 3.2.17, the image of j^* is precisely $W_2 H^2(U)$. So it suffices to prove that $\text{Im } j_! + \ker j^* = H^2(X)$. By exactness, this is equivalent to prove that $\text{Im } i_* + \ker i^* = H^2(X)$. Therefore, it suffices to prove that $\psi = i^* i_*$ is an isomorphism. In fact, the morphism ψ maps $\xi \in H_T^2(X) \cong H_2(T)$ to $\xi^\dagger : H_2(T) \rightarrow \mathbb{Q}$, where $\xi^\dagger(\gamma) = \int_X \xi \cup i_*(\gamma)$. This defines a symmetric bilinear form on $H_2(T)$ defined by the intersection number of components of T viewed in X . To show that ψ is an isomorphism, it suffices to show that this bilinear form is nondegenerate. In the case where T is a triangle, then the intersection matrix is

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

In the case when T is a nodal \mathbb{P}^1 , since it is cut out by hyperplane section away from the singularities of the weighted projective space, so it is an ample divisor, therefore the self intersection of T is nonzero. \square

Theorem 10.3. *Let $M_D = S \rightarrow \mathbb{C}$ be any of our five cases. Let M_B be the corresponding character varieties. Then perverse filtration of M_D and mixed Hodge filtration of M_B correspond.*

Proof. By the spectral sequence of weight filtration of mixed Hodge structure, dimensions of graded pieces of weight filtration is easily computed.

$$\dim \mathrm{Gr}_w^W H^d(M_B) = \begin{cases} 1 & w = d = 0 \\ 1 & w = 4, d = 2 \\ 2 & w = 2, d = 1, \mathbb{C}^* \times \mathbb{C}^* \text{ case} \\ k & w = 2, d = 2, \text{other 4 cases} \\ 0 & \text{otherwise.} \end{cases}$$

Compare with proposition 9.2, numerical $P = W$ holds for $n = 1$ in our five cases. To prove the full version of the full $P = W$ conjecture, it suffices to prove that $P_1 H^2(M_D) = W_2 H^2(M_B)$ in $\widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ cases. (\widetilde{A}_0 cases is trivially true.) By proposition 9.2, $P_1 H^2(M_D)$ in four cases are all spanned by the fundamental classes of exceptional curves of the minimal resolutions, and $H_c^2(M_D)$ is generated by the fundamental classes of exceptional curves and a generic fiber of the morphism $M_D \rightarrow \mathbb{C}$. Note that the fiber class is 0 in $H^2(M_D)$, so we have

$$P_1 H^2(M_D) = \mathrm{Im}(H_c^2(M_D) \rightarrow H^2(M_D)).$$

By lemma 10.2, we have $W_2 H^2(M_B) = \mathrm{Im}(H_c^2(M_B) \rightarrow H^2(M_B))$. This completes the proof. \square

Remark 10.4. In [5], theorem 3.2.1 asserts that the forgetful map $H_c^{6g-6}(M_D) \rightarrow H^{6g-6}(M_D)$ is zero map, where $6g-6$ is the complex dimension of the moduli space in the context. In fact, they consider moduli space of degree 1 Higgs bundle and twisted representations, so their result does not contradict ours.

11. PERVERSE NUMBERS AND NUMERICAL $P = W$

In this section, we give some partial numerical evidence for $P = W$ conjecture in our five families of Hitchin fibration. First, we use proposition 6.2 and proposition 9.2 to compute the perverse numbers $\mathrm{Gr}_\bullet H^*(S^{[n]})$, where S is any of our five cases defined in definition 9.1. Then we state the moduli description of corresponding character varieties according to Simpson's non-abelian Hodge theory. On the contrary to the case $n = 1$ described in last section, the character varieties for $n > 1$ are still unknown. Conjecture 1.2.1 in [10] predicts that the mixed Hodge numbers of character varieties can be computed by a combinatorial formula. We made a conjecture that in our five families of Hitchin fibration, the perverse numbers equal the conjectural mix Hodge numbers of corresponding character varieties. We use Mathematica to prove our conjecture for small n .

Theorem 11.1. *Let $f : S \rightarrow \mathbb{C}$ be any of our five cases. Denote perverse numbers by $p^{i,j} = \dim \mathrm{Gr}_i H^j(S^{[n]})$. Let perverse Poincaré polynomial be $P_n(q, t) = \sum_{i,j} p^{i,j} q^i t^j$. Then for \widetilde{A}_0 case, the generating series is*

$$\sum_{n=0}^{\infty} s^n P_n(q, t) = \prod_{m=1}^{\infty} \frac{(1 + s^m q^m t^{2m-1})^2}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^{m+1} t^{2m})}.$$

For the other four cases, the generating series are

$$\sum_{n=0}^{\infty} s^n P_n(q, t) = \prod_{m=1}^{\infty} \frac{1}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^m t^{2m})^k (1 - s^m q^{m+1} t^{2m})}$$

where k is defined in proposition 9.2.

Proof. We prove the equalities by expanding both hand sides and identify the corresponding terms. Since all cases are similar, we prove \widehat{D}_4 case for example. By Künneth formula and MacDonal theorem, we have

$$\begin{aligned}
& H^*(S^{[n]})[n] \\
&= \mathbb{H} \left(\bigoplus_{\nu=1^{a_1} \dots n^{a_n}} Rr_{C,*}^{(\nu)} \left(\bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} \mathcal{P}_0^{\{\mathbf{r}\}} \boxtimes \mathcal{P}_1^{(\mathbf{s})} \boxtimes \mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})] \right) \\
&= \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} \mathbb{H} \left(\mathcal{P}_0^{\{\mathbf{r}\}} \boxtimes \mathcal{P}_1^{(\mathbf{s})} \boxtimes \mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})] \\
&= \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} \mathbb{H} \left(\mathcal{P}_0^{\{\mathbf{r}\}} \right) \otimes \mathbb{H} \left(\mathcal{P}_1^{(\mathbf{s})} \right) \otimes \mathbb{H} \left(\mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})]
\end{aligned}$$

By proposition 9.2, we have $\mathcal{P}_0 = \mathbb{Q}_{\mathbb{C}}[1]$, $\mathcal{P}_1 = j_* \hat{R}^1[1] \oplus \mathbb{Q}_0^4$, $\mathcal{P}_2 = \mathbb{Q}_{\mathbb{C}}[1]$. Note that $j_* \hat{R}^1$ has no cohomology, we have

	$\mathbb{H}(\mathcal{P}_0)$	$\mathbb{H}(\mathcal{P}_1)$	$\mathbb{H}(\mathcal{P}_2)$
dimension	1	4	1
degree	-1	0	-1

Together with proposition 3.1 and remark 4.4, the table shows that the summand

$$\mathbb{H} \left(\mathcal{P}_0^{\{\mathbf{r}\}} \right) \otimes \mathbb{H} \left(\mathcal{P}_1^{(\mathbf{s})} \right) \otimes \mathbb{H} \left(\mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})]$$

is of dimension $\prod_{i=1}^n \binom{s_i + 3}{3}$ as a vector space, and all cohomology classes in it are of degree $\sum_{i=1}^n -r_i + s_i + t_i$ and perversity $\sum_{i=1}^n s_i + 2t_i$. (Here we use the fact that $\mathbb{H}(\mathcal{P}_0)$ and $\mathbb{H}(\mathcal{P}_2)$ are in odd degree, so by our convention the alternating products are in fact symmetric product if they are just viewed as vector spaces.) So

$$\begin{aligned}
P_n(q, t) &= \sum_{\nu=1^{a_1} \dots n^{a_n}} q^{n-l(\nu)} t^{2n-l(\nu)} \prod_{i=1}^n \sum_{r_i+s_i+t_i=a_i} \binom{s_i + 3}{3} q^{s_i+2t_i} t^{-r_i+s_i+t_i} \\
&= \sum_{\nu=1^{a_1} \dots n^{a_n}} q^{n-l(\nu)} t^{2n-l(\nu)} \prod_{i=1}^n t^{-a_i} \sum_{r_i+s_i+t_i=a_i} \binom{s_i + 3}{3} q^{s_i+2t_i} t^{2s_i+2t_i} \\
&= \sum_{\nu=1^{a_1} \dots n^{a_n}} q^{n-l(\nu)} t^{2n-2l(\nu)} \prod_{i=1}^n \sum_{r_i+s_i+t_i=a_i} \binom{s_i + 3}{3} q^{s_i+2t_i} t^{2s_i+2t_i}
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} P_n(q, t) = \sum_{n=0}^{\infty} \sum_{\nu=1^{a_1} \dots n^{a_n}} q^{n-l(\nu)} t^{2n-2l(\nu)} \prod_{i=1}^n \sum_{r_i+s_i+t_i=a_i} \binom{s_i + 3}{3} q^{s_i+2t_i} t^{2s_i+2t_i}$$

On the other hand, to get a term in the product right hand side with factor s^n is equivalent to the following data. (1) A partition $\nu = 1^{a_1} \dots n^{a_n}$ of n , such that the factor $m = i$ contributes $(s^i)^{a_i}$, (2) a triple (r_i, s_i, t_i) for each i satisfying $r_i + s_i + t_i = a_i$, such that the expansions of three parenthesis contribute $(s^i)^{r_i}$,

$(s^i)^{s_i}$ and $(s^i)^{t_i}$, respectively. So the term obtained in this way is

$$\begin{aligned}
& \prod_{i=1}^n \binom{s_i + 3}{3} s^{i(r_i + s_i + t_i)} q^{(i-1)r_i + is_i + (i+1)t_i} t^{(2i-2)r_i + 2is_i + 2it_i} \\
&= \prod_{i=1}^n \binom{s_i + 3}{3} s^{ia_i} q^{(i-1)a_i + s_i + 2t_i} t^{(2i-2)a_i + 2s_i + 2t_i} \\
&= s^n q^{n-l(\nu)} t^{2n-2l(\nu)} \prod_{i=1}^n \binom{s_i + 3}{3} q^{s_i + 2t_i} t^{2s_i + 2t_i}
\end{aligned}$$

Here we use the fact that $a_i = r_i + s_i + t_i$, $n = \sum_{i=1}^n ia_i$ and $l(\nu) = \sum_{i=1}^n a_i$. Compare with the expansion of left hand side, the theorem follows. \square

In parabolic non-abelian Hodge theory, the moduli of parabolic Higgs bundle is canonically diffeomorphic to the corresponding character variety. The $P = W$ conjecture asserts that under this canonical diffeomorphism, the weight filtration in mixed Hodge filtration on cohomology of character variety corresponds to the perverse filtration on cohomology of Higgs moduli with respect to the Hitchin map. By the Simpson's table on page 720 in [15], we may find the character varieties correspond to our five families of moduli of parabolic Higgs bundles. The conjecture 1.2.1 in [10] conjectures that all cohomology class are of Hodge-Tate type and the mixed Hodge numbers depend on the multiplicities of eigenvalues of the monodromy action around the punctures rather than the eigenvalues themselves. So for our purpose, we list the corresponding moduli description of character varieties for our five families of Hitchin systems without mentioning the specific eigenvalues for the monodromy action.

- (1) Let E be any elliptic curve. Consider $GL(n, \mathbb{C})$ -representation of $\pi_1(E \setminus p)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalue are of type

$$(n-1, 1).$$

- (2) Let $C = \mathbb{P}^1 \setminus \{p_1, \dots, p_4\}$. Consider $GL(2n, \mathbb{C})$ -representation of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalue are of type

$$(n, n)(n, n)(n, n)(n, n-1, 1).$$

- (3) Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(3n, \mathbb{C})$ -representation of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalue are of type

$$(n, n, n)(n, n, n)(n, n, n-1, 1).$$

- (4) Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(4n, \mathbb{C})$ -representation of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalue are of type

$$(2n, 2n)(n, n, n, n)(n, n, n, n-1, 1).$$

- (5) Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(6n, \mathbb{C})$ -representation of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalue are of type

$$(3n, 3n)(2n, 2n, 2n)(n, n, n, n, n, n-1, 1).$$

So if we believe $P = W$, the perverse numbers of Hitchin system should equal the mixed Hodge numbers of corresponding character varieties. In fact, the conjecture 1.2.1 in [10] and theorem 11.1 yield to the following conjecture.

Conjecture 11.2. *Let μ be multi-partition defined above. Let k be defined in proposition 9.2. Let \mathbb{H}_μ be defined as in section 1.1 of [10]. Then we have the following identities. For \widetilde{A}_0 case, we have*

$$\prod_{m=1}^{\infty} \frac{(1 + s^m q^m t^{2m-1})^2}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^{m+1} t^{2m})} = \sum_{n=0}^{\infty} (st^2 q)^n \mathbb{H}_\mu(-\sqrt{q}, \frac{\sqrt{q}}{t})$$

For the other four cases, we have

$$\prod_{m=1}^{\infty} \frac{1}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^m t^{2m})^k (1 - s^m q^{m+1} t^{2m})} = \sum_{n=0}^{\infty} (st^2 q)^n \mathbb{H}_\mu(-\sqrt{q}, \frac{\sqrt{q}}{t})$$

Remark 11.3. In fact, the conjecture for \widetilde{A}_0 case is the cohomology version of conjecture 4.2.1 in [11], and the other four cases are new. With the help of Mathematica, we prove for $n \leq 6$ for \widetilde{D}_4 case, $n \leq 4$ for \widetilde{E}_6 case, $n \leq 3$ for \widetilde{E}_7 and $n \leq 2$ for \widetilde{E}_8 case. Efforts to prove for general n are so far all ended with combinatorial difficulties. More understanding on q, t -Kostka numbers would be helpful.

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